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SOLUTIONS OF HIGHER ORDER SINGULAR NONLINEAR $(l-1,1)$ CONJUGATE-TYPE FRACTIONAL DIFFERENTIAL EQUATIONS WITH LOWER-UPPER SOLUTION

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ABSTRACT. Using the monotone iterative technique coupled with method of lower and upper solutions, we establish the existence and uniqueness of solutions for higher order singular nonlinear $(l-1,1)$ conjugate-type fractional differential equation with one nonlocal term.

KEYWORDS: Monotone iterative technique, lower-upper solutions, integral boundary conditions, fractional differential equations, existence and uniqueness solution, Banach contraction principle.
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1. Introduction

Fractional differential equations are the generalization of ordinary differential equations to non-integer order. This generalization has interesting applications in various fields of chemistry, physics, mechanics, economics, electrodynamics etc. [4, 10]. Boundary value problems [BVP] of fractional differential equations have widespread attention and some attractive results obtained [1, 7, 9, 19] recently. Monotone iterative technique plays an important role to obtain existence of solutions of nonlinear fractional differential equations [5]. This technique is used to obtain the solutions of nonlinear initial value problems [6], boundary value problems [2, 14, 19, 21]. Existence and uniqueness of solutions of Riemann-Liouville fractional

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differential equations with integral boundary conditions is obtained by Nanware et. al. [8]. Sun and Zhao [12] studied the fractional differential equations with integral boundary conditions using monotone iterative method.

In the recent years, the theory of singular boundary value problems has become an important area of investigation [3, 13, 17, 18]. The existence of solutions by using various methods such as lower and upper solution method and fixed point theorem is proved. In [20] X. Zhang et al. obtained the existence and uniqueness of positive solutions when g has singularities at $r = 0$ and (or) 1 by using monotone iterative method. In 2020 [11] S. Song et al. investigated the existence of extremal solutions by using monotone iterative technique coupled with lower and upper solutions for the problem

$$\begin{cases} -D_{0+}^{\nu} z(r) = g(r, z(r)), \quad r \in [0,1], \\ z(0) = 0, \qquad z(1) = \displaystyle\int_0^1 z(s)\, d\eta(s), \end{cases}$$

where $1 < \nu < 2$, D_{0+}^{ν} is the Riemann-Liouville fractional derivative and $\eta(r)$ is a positive measure function. Y. Wang et al. [15] studied the positive properties of the green function for the Dirichlet-type problem

$$\begin{cases} -D_{0+}^{\nu} z(r) + az(r) = g(r, z(r)), \quad 0 < r < 1, \\ z(0) = 0, \qquad z(1) = 0, \end{cases}$$

where $1 < \nu < 2$, $a > 0$, D_{0+}^{ν} is the Riemann-Liouville fractional derivative. Y. Wang et al.[16] established the existence of positive solutions for resonant problem.

Inspired by the aforementioned works, in this paper we give some sufficient conditions, under which following problems have extremal solutions

$$\begin{cases} D_{0+}^{\nu} z(r) + g(r, z(r)) = 0, \quad 0 < r < 1, \quad l - 1 < \nu \le l, \\ z^{(k)}(0) = 0, \quad 0 \le k \le l-2, \quad z(1) = \displaystyle\int_0^1 z(s)\, d\eta(s), \end{cases} \tag{1.1}$$

where, D_{0+}^{ν} is the Riemann-Liouville fractional derivative of order $l \ge 2$, $l \in \mathbb{N}$, g has singularities at $r = 0$ and (or) 1, η is a function of bounded variation and $\displaystyle\int_0^1 z(s) d\eta(s)$ denotes the Riemann-Stieltjes integral of z with respect to η, $d\eta$ can be signed measure . The layout of this paper is as follows: In section 2, we present some basic definitions and lemmas that will be used to prove our main results. Section 3 is devoted to uniqueness of solution to BVP (1.1) by using Banach contraction principle. In Section 4, we developed the monotone iterative method and applied it to obtain existence and uniqueness results for Riemann-Liouville fractional differential equations with integral boundary conditions.

2. Preliminaries

In this section, we present some useful definitions and lemmas that will be used in the next section to attain existence and uniqueness results for the nonlinear of BVP (1.1).

Definition 2.1. [10] The Riemann-Liouville fractional integral of order $\nu > 0$ of a function $z : (0, \infty) \to \mathbb{R}$ is given by

$$I^{\nu} z(r) = \frac{1}{\Gamma(\nu)} \int_0^r (r - s)^{\nu - 1} z(s)\, ds$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2. [10] The Riemann-Liouville fractional derivative of order $\nu > 0$ of a function $z:(0,\infty)\to\mathbb{R}$ is given by

$$D^{\nu}z(r)=\frac{1}{\Gamma(l-\nu)}(\frac{d}{dr})^{l}\int_0^r (r-s)^{\nu-l+1}z(s)\,ds$$

where $l\in\mathbb{N}$ as the unique positive integer satisfying $l-1<\nu\le l$, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.3. A function $\dot{x}_0\in C([0,\,1])$ is called a lower solution of BVP (1.1) if it satisfies

$$\begin{cases} D^{\nu}_{0+}\dot{x}_0(r)+g(r,\dot{x}_0(r))\ge 0, \quad 0<r<1, l-1<\nu\le l,\\ \dot{x}_0^{(k)}(0)=0, \quad 0\le k\le l-2, \qquad \dot{x}_0(1)\le \int_0^1 \dot{x}_0(s)\,d\eta(s). \end{cases} \tag{2.1}$$

Definition 2.4. A function $\dot{y}_0\in C([0,\,1])$ is called a upper solution of BVP (1.1) if it satisfies

$$\begin{cases} D^{\eta}_{0+}\dot{y}_0(r)+g(r,\dot{y}_0(r))\le 0, \quad 0<r<1, \quad l-1<\nu\le l,\\ \dot{y}_0^{(k)}(0)=0, \quad 0\le k\le l-2, \qquad \dot{y}_0(1)\ge \int_0^1 \dot{y}_0(s)\,d\eta(s). \end{cases} \tag{2.2}$$

Denote

$$\rho(r)=\frac{\nu-2}{\Gamma(\nu-1)}+\sum_{k=1}^{\infty}\frac{r^k}{\Gamma((k+1)\nu-2)}.$$

It is easy to check that (see [15, 16])

$$\rho(0)=\frac{\nu-2}{\Gamma(\nu-1)}<0,$$

$$\rho^{'}(r)=\sum_{k=1}^{\infty}\frac{kr^{k-1}}{\Gamma((k+1)\nu-2)}>0, \text{ on } \quad (0,\infty)$$

and

$$\underset{r\to+\infty}{lim}\rho(r)=+\infty.$$

Therefore, there exist a unique $a^*>0$ such that $\quad \rho(a^*)=0$.
Set

$$G_a(r)=r^{\nu-1}E_{\nu,\,\nu}(ar^{\nu}), \quad \text{where} \quad E_{\nu,\,\nu}(r)=\sum_{k=0}^{\infty}\frac{r^k}{\Gamma((k+1)\nu)} \tag{2.3}$$

is the Mittag-Leffler function ([4, 10]).
For convenience, we list here the following assumptions.
B1] the parameter a satisfies $a\in(0,a^*]$,
B2] $\eta(r)$ is bounded variation in $(0,1)$ such that $0<\alpha\le 1$, $\alpha=\int_0^1 G_a(s)\,d\eta(s)$ and $0\le\zeta_\eta(s)=\int_0^1 H_a(r,s)\,d\eta(s)$, $0<G_a(1)-\int_0^1 G_a(s)\,d\eta(s)$,
B3] $g\in C((0,1)\times[0,\infty),[0,\infty))$ and

$g(r,u)-g(r,v)\ge -a(u-v)$ for $\dot{x}_0\le u\le v\le \dot{y}_0$, $r\in(0,1)$.
Set

$$K_a(r,s)=H_a(r,s)+G_a(r)h^*(s)$$

where,

$$h^*(s) = \frac{\zeta_\eta(s)}{G_a(1) - \alpha},$$

$$H_a(r,s) = \frac{1}{G_a(1)} \begin{cases} G_a(r)G_a(1-s) - G_a(r-s)G_a(1) & \text{if } 0 \le s \le r \le 1 \\ G_a(r)G_a(1-s) & \text{if } 0 \le r \le s \le 1. \end{cases} \tag{2.4}$$

Lemma 2.5. [15] *Suppose that* $[B1]$ *holds and* $y \in L[0,1]$. *Then the problem*

$$\begin{cases} -D^\nu_{0+} z(r) + az(r) = q(r), & 0 < r < 1, \\ z(0) = 0, & z(1) = 0, \end{cases}$$

has a unique solutions

$$z(r) = \int_0^1 H_a(r,s)q(s)\,ds,$$

where

$$H_a(r,s) = \frac{1}{G_a(1)} \begin{cases} G_a(r)G_a(1-s) - G_a(r-s)G_a(1) & \text{if } 0 \le s \le r \le 1 \\ G_a(r)G_a(1-s) & \text{if } 0 \le r \le s \le 1. \end{cases}$$

Lemma 2.6. *Suppose that* $[B1]$, $[B2]$ *hold and* $y \in C([0,1])$. *Then linear fractional boundary value problem*

$$\begin{cases} D^\nu_{0+} z(r) - az(r) + q(r) = 0, & 0 < r < 1, \quad l-1 < \nu \le l, \\ z^{(k)}(0) = 0, \quad 0 \le k \le l-2, & z(1) = \int_0^1 z(s)\,d\eta(s), \end{cases} \tag{2.5}$$

has the following unique solution

$$z(r) = \int_0^1 K_a(r,s)q(s)\,ds.$$

Proof:- First apply I^ν on linear equation (2.5) and using result, see in [4, 10], we get

$$z(r) = -\int_0^r G_a(r-s)q(s)\,ds + C_0G_a(r) + C_1G_a'(r) + C_2G_a''(r) \ldots + C_{l-1}G_a^{(l-1)}(r). \tag{2.6}$$

Since $z(0) = 0$ then $C_{l-1} = 0$. Similarly

$$z'(0) = z''(0) = \ldots = z^{l-2}(0) = 0$$

gives

$$C_1 = C_2 = \ldots = C_{l-2} = 0.$$

Then equation (2.6) becomes

$$z(r) = -\int_0^r G_a(r-s)q(s)\,ds + C_0G(r).$$

Using $z(1) = \int_0^1 z(s)\,d\eta(s)$, we obtain

$$C_0 = \frac{1}{G_a(1)}\left[\int_0^1 z(s)\,d\eta(s) + \int_0^1 G_a(1-s)q(s)\,ds\right].$$

Hence,

$$z(r) = -\int_0^r G_a(r-s)q(s)\,ds + \frac{G_a(r)}{G_a(1)}\left[\int_0^1 z(s)\,d\eta(s) + \int_0^1 G_a(1-s)q(s)\,ds\right],$$

$$
\begin{aligned}
&= -\int_0^r G_a(r-s)q(s)\,ds + \frac{G_a(r)}{G_a(1)}\int_0^1 z(s)\,d\eta(s) + \frac{G_a(r)}{G_a(1)}\int_0^r G_a(1-s)q(s)\,ds \\
&\quad + \frac{G_a(r)}{G_a(1)}\int_r^1 G_a(1-s)q(s)\,ds, \\
&= \frac{1}{G_a(1)}\int_0^r \left[G_a(r)G_a(1-s) - G_a(1)G_a(r-s)\right]q(s)\,ds \\
&\quad + \frac{1}{G_a(1)}\int_r^1 G_a(r)G_a(1-s)q(s)\,ds + \frac{G_a(r)}{G_a(1)}\int_0^1 z(s)\,d\eta(s), \\
&= \int_0^1 H_a(r,s)q(s)\,ds + \frac{G_a(r)}{G_a(1)}\int_0^1 z(s)\,d\eta(s).
\end{aligned}
$$

Let,

$$
\int_0^1 z(s)\,d\eta(s) = \int_0^1 [\int_0^1 H_a(s,\tau)q(\tau)\,d\tau]d\eta(s) + \int_0^1 \frac{G_a(s)}{G_a(1)}\,d\eta(s)\int_0^1 z(s)\,d\eta(s).
$$

Therefore

$$
\begin{aligned}
[1 - \frac{1}{G_a(1)}\int_0^1 G_a(s)\,d\eta(s)]\int_0^1 z(s)\,d\eta(s) &= \int_0^1 [\int_0^1 H_a(s,\tau)q(\tau)\,d\tau]d\eta(s) \\
\int_0^1 z(s)\,d\eta(s) &= \frac{1}{[1 - \frac{1}{G_a(1)}\int_0^1 G_a(s)\,d\eta(s)]}\int_0^1\int_0^1 H_a(s,\tau)q(\tau)\,d\tau d\eta(s).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
z(r) &= \int_0^1 H_a(r,s)q(s)\,ds + \frac{G_a(r)}{G_a(1)}\left[\frac{1}{1 - \frac{1}{G_a(1)}\int_0^1 G_a(s)\,d\eta(s)}\right] \\
&\qquad \int_0^1\int_0^1 H_a(s,\tau)q(\tau)\,d\tau d\eta(s), \\
&= \int_0^1 H_a(r,s)q(s)\,ds + \frac{G_a(r)}{G_a(1) - \int_0^1 G_a(s)\,d\eta(s)}\int_0^1\int_0^1 H_a(s,\tau)q(\tau)\,d\tau d\eta(s), \\
&= \int_0^1 \left[H_a(r,s) + \frac{G_a(r)}{G_a(1) - \int_0^1 G_a(s)\,d\eta(s)}\int_0^1 H_a(s,\tau)d\eta(\tau)\right]q(s)\,ds, \\
z(r) &= \int_0^1 K_a(r,s)q(s)\,ds.
\end{aligned}
$$

Lemma 2.7. *Suppose* $[B1]$, $[B2]$ *holds,then the function* $K_a(r,s)$ *has the following properties*

(i) $K_a(r,s) > 0 \quad \forall\ r,s \in (0,1)$,
(ii) $\psi_2(s)r^{\nu-1} \le K_a(r,s) \le \psi_1(s)r^{\nu-1}, \quad \forall\ r,s \in (0,1)$

where, $\psi_1(s) = G_a(1-s) + G_a(1)h^*(s), \quad \psi_2(s) = \frac{1}{\Gamma(\nu)}h^*(s)$.

Proof:- We need to prove that (2) holds. By equation (2.7)

$$\frac{r^{\nu-1}}{\Gamma(\nu)} \le G_a(r) = r^{\nu-1}\sum_{k=0}^{\infty}\frac{a^k r^{\nu k}}{\Gamma((k+1)\nu)} \le r^{\nu-1}G_a(1), \quad r\in(0,1). \tag{2.7}$$

$$G_a^{'}(r) = \sum_{k=0}^{\infty}\frac{a^k r^{(k+1)\nu-2}}{\Gamma((k+1)\nu-1)} > 0, \quad r\in(0,1) \tag{2.8}$$

$$\begin{aligned}
G_a^{''}(r) &= \sum_{k=0}^{\infty}\frac{a^k[(k+1)\nu-2]r^{(k+1)\nu-3}}{\Gamma((k+1)\nu-1)},\\
&= r^{\nu-3}\sum_{k=0}^{\infty}\frac{a^k[(k+1)\nu-2]r^{k\nu}}{\Gamma((k+1)\nu-1)},\\
&= r^{\nu-3}\left[\frac{\nu-2}{\Gamma(\nu-1)} + \sum_{k=1}^{\infty}\frac{a^k r^{k\nu}[(k+1)\nu-2]}{\Gamma((k+1)\nu-1)}\right],\\
&= r^{\nu-3}\left[\frac{\nu-2}{\Gamma(\nu-1)} + \sum_{k=1}^{\infty}\frac{a^k r^{k\nu}}{\Gamma((k+1)\nu-2)}\right],\\
&= r^{\nu-3}\rho(ar^{\nu}) < r^{\nu-3}\rho(a) \le r^{\nu-3}\rho(a^*) = 0, \quad r\in(0,1),
\end{aligned} \tag{2.9}$$

which implies that $G_a(r)$ is strictly increasing on $(0,1)$ and $G_a^{'}(r)$ is strictly decreasing on $(0,1)$. Therefore by (2.7) we have,

$$\begin{aligned}
K_a(r,s) &= H_a(r,s) + G_a(r)h^*(s),\\
&\le \frac{G_a(r)G_a(1-s)}{G_a(1)} + G_a(r)h^*(s),\\
&= \left[\frac{G_a(1-s)}{G_a(1)} + h^*(s)\right]G_a(r),\\
&\le \left[\frac{G_a(1-s)}{G_a(1)} + h^*(s)\right]r^{\nu-1}G_a(1),\\
&= [G_a(1-s) + G_a(1)h^*(s)]r^{\nu-1},\\
&= \psi_1(s)r^{\nu-1},
\end{aligned} \tag{2.10}$$

where, $\psi_1(s) = G_a(1-s) + G_a(1)h^*(s)$.
On the other hand, when $0 \le r \le s \le 1$. Note that $G_a(0) = 0$ and monotonocity of $G_a(r)$, it is clear that

$$G_a(r)G_a(1-s) > 0. \tag{2.11}$$

Hence $H_a(r,s) > 0$ and also by $[B2]$, $K_a(r,s) > 0$ when $0 < r \le s \le 1$.
When $0 < s < r < 1$, we have

$$\begin{aligned}
\frac{\partial}{\partial s}[G_a(r)G_a(1-s) - G_a(r-s)G_a(1)] &= -G_a(r)G_a^{'}(1-s) + G_a(1)G_a^{'}(r-s)\\
&\ge [G_a(1) - G_a(r)]G_a^{'}(1-s).
\end{aligned} \tag{2.12}$$

Integrating with respect to s, we obtain

$$G_a(r)G_a(1-s) - G_a(r-s)G_a(1) \ge \int_0^s [G_a(1) - G_a(r)]G_a^{'}(1-\mu)\,d\mu,$$

$$= [G_a(1) - G_a(r)] \left[\frac{G_a(1-\mu)}{-1} \right]_0^s,$$
$$= [G_a(1) - G_a(r)][G_a(1) - G_a(1-s)] > 0. \tag{2.13}$$

Then, by (2.4), (2.11), (2.13), we get

$$H_a(r,s) = G_a(r)G_a(1-s) - G_a(r-s)G_a(1) > 0 \quad r,s \in (0,1).$$

Now,

$$K_a(r,s) = H_a(r,s) + G_a(t)h^*(s) \geq G_a(r)h^*(s),$$
$$\geq \frac{r^{\nu-1}}{\Gamma(\nu)}h^*(s) = \psi_2(s)r^{\nu-1} > 0 \quad r,s \in (0,1)$$

where, $\psi_2(s) = \frac{1}{\Gamma(\nu)}h^*(s)$. Hence the proof.

Lemma 2.8. *For* $0 < r_1 \leq r_2 < 1$

(i) $|G_a(r_2) - G_a(r_1)| < E_{\nu,\nu-1}(a)\,|r_2 - r_1| = G_a(1)|r_2 - r_1|$,
(ii) $|G_a(r_2-s) - G_a(r_1-s)| < E_{\nu,\nu-1}(a)\,|r_2 - r_1| = G_a(1)|r_2 - r_1|$,
(iii) $|H_a(r_2,s) - H_a(r_1,s)| < 2[G_a(1)]^2|r_2 - r_1|$,
(iv) $|K_a(r_2,s) - K_a(r_1,s)| \leq \max\limits_{0\leq s\leq 1}|K_a(r_2,s) - K_a(r_1,s)| < G_a(1)[2G_a(1) + |h^*(s)|]|r_2 - r_1|$.

Proof:-

1] $$|G_a(r_2) - G_a(r_1)| = \left| r_2^{\nu-1}E_{\nu,\nu}(ar_2^\nu) - r_1^{\nu-1}E_{\nu,\nu}(ar_1^\nu) \right|,$$
$$= \left| r_2^{\nu-1}\sum_{k=o}^{\infty}\frac{a^k r_2^{\nu k}}{\Gamma((k+1)\nu)} - r_1^{\nu-1}\sum_{k=o}^{\infty}\frac{a^k r_2^{\nu k}}{\Gamma((k+1)\nu)} \right|,$$
$$= \sum_{k=o}^{\infty}\frac{a^k}{\Gamma((k+1)\nu)}\left| r_2^{\nu(k+1)-1} - r_1^{\nu(k+1)-1} \right|.$$

Applying mean value theorem, we get

$$r_2^{\nu(k+1)-1} - r_1^{\nu(k+1)-1} < [\nu(k+1)-1](r_2 - r_1).$$

Therefore

$$|G_a(r_2) - G_a(r_1)| < \sum_{k=o}^{\infty}\frac{a^k[\nu(k+1)-1]}{\Gamma((k+1)\nu)}|r_2 - r_1|,$$
$$= \sum_{k=o}^{\infty}\frac{a^k}{\Gamma((k+1)\nu-1)}|r_2 - r_1|,$$
$$= E_{\nu,\nu-1}(a)|r_2 - r_1| = G_a(1)|r_2 - r_1|.$$

2] $$|G_a(r_2-s) - G_a(r_1-s)| = \left| (r_2-s)^{\nu-1}E_{\nu,\nu}(a(r_2-s)^\nu) - (r_1-s)^{\nu-1}E_{\nu,\nu}(a(r_1-s)^\nu) \right|,$$
$$= \left| (r_2-s)^{\nu-1}\sum_{k=o}^{\infty}\frac{a^k(r_2-s)^{\nu k}}{\Gamma((k+1)\nu)} - (r_1-s)^{\nu-1}\sum_{k=o}^{\infty}\frac{a^k(r_2-s)^{\nu k}}{\Gamma((k+1)\nu)} \right|,$$
$$= \sum_{k=o}^{\infty}\frac{a^k}{\Gamma((k+1)\nu)}\left| (r_2-s)^{\nu(k+1)-1} - (r_1-s)^{\nu(k+1)-1} \right|.$$

Applying mean value theorem, we get

$$(r_2-s)^{\nu(k+1)-1} - (r_1-s)^{\nu(k+1)-1} < [\nu(k+1)-1](r_2 - r_1).$$

Therefore

$$\begin{aligned}|G_a(r_2-s)-G_a(r_1-s)| &< \sum_{k=o}^{\infty}\frac{a^k}{\Gamma((k+1)\nu-1)}|r_2-r_1| \\ &= E_{\nu,\,\nu-1}(a)|r_2-r_1| = G_a|r_2-r_1|\end{aligned}$$

3] $$\begin{aligned}|H_a(r_2,s)-H_a(r_1,s)| &= |[G_a(r_2)G_a(1-s)-G_a(1)G_a(r_2-s)] \\ &\quad -[G_a(r_1)G_a(1-s)-G_a(1)G_a(r_1-s)]|, \\ &< G_a(1-s)|G_a(r_2)-G_a(r_1)|+G_a(1)|G_a(r_1-s)-G_a(r_2-s)|, \\ &< G_a(1-s)E_{\nu,\,\nu}(a)|r_2-r_1|+G_a(1)E_{\nu,\,\nu}(a)|r_2-r_1|, \\ &= E_{\nu,\,\nu}(a)[G_a(1-s)+G_a(1)]|r_2-r_1|, \\ &= G_a(1)[G_a(1-s)+G_a(1)]|r_2-r_1|, \\ &< 2[G_a(1)]^2|r_2-r_1|.\end{aligned}$$

4] $$\begin{aligned}|K_a(r_2,s)-K_a(r_1,s)| &\le \max_{0\le s\le 1}|K_a(r_2,s)-K_a(r_1,s)|, \\ &= \max_{0\le s\le 1}|[H_a(t_2,s)-H_a(t_1,s)]+[G_a(t_2)-G_a(r_1)]h^*(s)|, \\ &< 2[G_a(1)]^2|r_2-r_1|+G_a(1)|r_2-r_1||h^*(s)|, \\ &= G_a(1)[2G_a(1)+|h^*(s)|]|r_2-r_1|.\end{aligned}$$

Hence the proof.

3. Main Results

Let $\mathscr{C}=C([0,1])$ be endowed with the norm $\|z\|=\max_{0\le r\le 1}|z(r)|$, then $(\mathscr{C},\|.\|)$ is a Banach space. Now we define the operator $T:\mathscr{C}\to\mathscr{C}$ by

$$(Tz)(r)=\int_0^1 K_a(r,s)g(s,z(s))\,ds.$$

Theorem 3.1. *Prove $T:\mathscr{C}\to\mathscr{C}$ is uniformly continuous.*

Proof:- The operator $T:\mathscr{C}\to\mathscr{C}$ is continuous in the view of non-negativeness and continuity of $K_a(r,s)$, $H_a(r,s)$ and $g(r,z)$. Let $S\subset\mathscr{C}$ be bounded i.e. $\exists$ a positive constants $M>0$ such that $\|z\|<M\ \forall\ z\in S$, Let $L^*=\max_{0\le r\le 1}|g(r,z)|$ then by Lemma 2.7 the operator $T:S\to\mathscr{C}$ is bounded uniformly. Now to prove T(S) is equicontinuous.

If $z\in S$, $\quad 0<r_1<r_2<1$ then

$$\begin{aligned}|(Tz)(r_2)-(Tz)(r_1)| &= \left|\int_0^1[K_a(r_2,s)-K_a(r_1,s)]g(s,z(s))\,ds\right|, \\ &\le \max_{0\le s\le 1}\int_0^1|K_a(r_2,s)-K_a(r_1,s)|\,|g(s,z(s))|\,ds, \\ &< L^*G_a(1)|r_2-r_1|\int_0^1[2G_a(1)+|h^*(s)|]\,ds, \\ &< L^*G_a(1)|r_2-r_1|[2G_a(1)+\int_0^1|h^*(s)|]\,ds.\end{aligned}$$

Then $|(Tz)(r_2)-(Tz)(r_1)|\to 0$ uniformly as $r_1\to r_2$. This shows that $T(S)$ is equicontinuous on $\mathscr{C}$. Then by Arzela-Ascoli theorem, the operator $T:\mathscr{C}\to\mathscr{C}$ is completely continuous.

Theorem 3.2. *Assume that* $[B1]$, $[B2]$ *holds. Then there exist nonnegative constant* N^* *such that function* g *satisfies* $|g(r,x)-g(r,y)| \leq N^*|x-y|$, $\quad r \in (0,1)$, $\quad x, y \in \mathscr{C}$ *and let*
$\Lambda = \int_0^1 \psi(s)\,ds$ *then the BVP (1.1) has a unique fixed point.*

Proof:- For any $x, y \in \mathscr{C}$, $\quad r, s \in (0,1)$ and using Lemma 2.7

$$\begin{aligned}
\|Tx(r) - Ty(r)\| &= \max_{0\leq r\leq 1} |Tx(r) - Ty(r)|\,, \\
&= \max_{0\leq r\leq 1} \left| \int_0^1 K_a(r,s)[g(r,x) - g(r,y)]\,ds \right|, \\
&= \max_{0\leq r\leq 1} \int_0^1 |K_a(r,s)||g(r,x) - g(r,y)|\,ds, \\
&\leq r^{\nu-1} N^* \|x-y\| \int_0^1 \psi(s)\,ds, \\
&= r^{\nu-1} N^* \Lambda \|u - v\|\,.
\end{aligned}$$

Then by Banach contraction mapping theorem, T has a unique fixed point in $\mathscr{C}$, i.e. the BVP (1.1) has a unique solution. The proof is complete.

4. Monotone Iterative Method

In this section, we develop monotone iterative technique combined with the method of lower-upper solutions and we prove the existence and uniqueness theorem of solution for BVP (1.1). For $\dot{x}_0, \dot{y}_0 \in \mathscr{C}$ with $\dot{x}_0 \leq \dot{y}_0$ for $r \in (0,1)$, we denote

$$\Omega^* = [\dot{x}_0, \dot{y}_0] = \{z \in \mathscr{C} : \dot{x}_0 \leq z(r) \leq \dot{y}_0, r \in (0,1)\}\,.$$

Lemma 4.1. *Assume that* $[B1]$, $[B2]$ *holds and* $z \in \mathscr{C}$ *satisfies*

$$\begin{aligned}
&-D^\nu z(r) + az(r) \geq 0, \\
&z^{(k)}(0) = 0, \qquad z(1) \geq \int_0^1 z(s)\,d\eta(s)
\end{aligned} \tag{4.1}$$

then for $r \in (0,1)$, $z(r) \geq 0$.

Proof:- Let $q(r) = -D^\nu z(r) + az(r)$ and $d = z(1) - \int_0^1 z(s)\,d\eta(s)$. Then from equation (4.1), we have $q(r) \geq 0$, $d \geq 0$. Then by Lemma 2.6, the problem (2.5) has unique solution which can be expressed as

$$\begin{aligned}
z(r) &= \int_0^1 K_a(r,s)q(s)\,ds, \\
&= \int_0^1 H_a(r,s)q(s)\,ds + \left[\frac{G_a(r)}{G_a(1) - \int_0^1 G_a(s)\,d\eta(s)}\right] \int_0^1 H_a(r,s)\,d\eta(s)
\end{aligned}$$

where,

$$H_a(r,s) = \frac{1}{G_a(1)} \begin{cases} G_a(r)G_a(1-s) - G_a(r-s)G_a(1) & \text{if } 0 \leq s \leq r \leq 1 \\ G_a(r)G_a(1-s) & \text{if } 0 \leq r \leq s \leq 1. \end{cases}$$

Then by Lemma 2.7, $H_a(r,s) \geq 0$ and $K_a(r,s) \geq 0$ $\forall\, r, s \in (0,1)$. Hence $z(r) \geq 0$, $\forall\, r, s \in (0,1)$.

Theorem 4.2. *Suppose $[B1]$, $[B2]$, $[B3]$ holds, then there exist monotone iterative sequences $\{\dot{x}_m\}$, $\{\dot{y}_m\} \subset \Omega^*$ such that $\dot{x}_m \to \dot{x}$, $\dot{y}_m \to \dot{y}$ as $m \to \infty$ uniformly in Ω^* and $\dot{x}$, $\dot{y}$ are a minimal and maximal solution of BVP (1.1) in Ω^* respectively.*

Proof:- For $\dot{x}_{m-1}$, $\dot{y}_{m-1} \in \mathscr{C}$, $m \geq 1$, we define two sequences $\{\dot{x}_m\}$, $\{\dot{y}_m\}$ respectively by the relations,

$$\begin{cases} D^{\nu}_{0+}\dot{x}_m(r) - a(\dot{x}_m(r) + \dot{x}_{m-1}(r)) + g(r, \dot{x}_{m-1}(r)) = 0, \quad 0 < r < 1 \\ \dot{x}_m^{(k)}(0) = 0, \qquad \dot{x}_m(1) = \displaystyle\int_0^1 \dot{x}_m(s)\, d\eta(s) \end{cases}$$

and

$$\begin{cases} D^{\nu}_{0+}\dot{y}_m(r) - a(\dot{y}_m(r) + \dot{y}_{m-1}(r)) + g(r, \dot{y}_{m-1}(r)) = 0, \quad 0 < r < 1, \\ \dot{y}_m^{(k)}(0) = 0, \qquad \dot{y}_m(1) = \displaystyle\int_0^1 \dot{y}_m(s)\, d\eta(s). \end{cases}$$

Then by Lemma 2.6, $\{\dot{x}_m\}$, $\{\dot{y}_m\}$ are well defined. Firstly we need to show that $\dot{x}_0(r) \leq \dot{x}_1(r) \leq \dot{y}_1(r) \leq \dot{y}_0(r)$ for any $r \in (0, 1)$.
Set $\dot{p}(r) = \dot{x}_1(r) - \dot{x}_0(r)$ and by definition of $\dot{x}_1(r)$ with lower solution $\dot{x}_0(r)$ we get,

$$\begin{aligned} -D^{\nu}_{0+}\dot{p}(r) + a\dot{p}(r) &= -D^{\nu}_{0+}(\dot{x}_1(r) - \dot{x}_0(r)) + a(\dot{x}_1(r) + \dot{x}_0(r)), \\ &= -D^{\nu}_{0+}\dot{x}_1(r) + a(\dot{x}_1(r) + \dot{x}_0(r)) + D^{\nu}_{0+}\dot{x}_0(r), \\ &\geq -a(\dot{x}_1(r) + \dot{x}_0(r)) + g(r, \dot{x}_0(r)) + a(\dot{x}_1(r) + \dot{x}_0(r)) - g(r, \dot{x}_0(r)) \\ &= 0. \\ Also, \dot{p}^{(k)}(0) &= \dot{x}_1^{(k)}(0) - \dot{x}_0^{k}(0) = 0, \\ \dot{p}(1) &= \dot{x}_1(1) - \dot{x}_0(1) \\ &\geq \int_0^1 \dot{x}_1(s)\, d\eta(s) - \int_0^1 x_0(s)\, d\eta(s) \\ &= \int_0^1 [\dot{x}_1(s) - \dot{x}_0(s)]\, d\eta(s) = \int_0^1 \dot{p}(s)\, d\eta(s). \end{aligned}$$

Then by Lemma 4.1, $\dot{p}(r) \geq 0 \Rightarrow \dot{x}_1(r) \geq \dot{x}_0(r)$, $r \in (0, 1)$.
Now to prove $\dot{y}_1(r) \leq \dot{y}_0(r)$ $\forall r \in (0, 1)$. For this, set $\dot{p}(r) = \dot{y}_1(r) - \dot{y}_0(r)$ and by definition of $\dot{y}_1(r)$ with upper solution $\dot{y}_0(r)$ we get,

$$\begin{aligned} -D^{\nu}_{0+}\dot{p}(r) + a\dot{p}(r) &= -D^{\nu}_{0+}(\dot{y}_1(r) - \dot{y}_0(r)) + a(\dot{y}_1(r) + \dot{y}_0(r)), \\ &= -D^{\nu}_{0+}\dot{y}_1(r) + a(\dot{y}_1(r) + \dot{y}_0(r)) + D^{\nu}_{0+}\dot{y}_0(r), \\ &\leq -a(\dot{y}_1(r) + \dot{y}_0(r)) + g(r, \dot{y}_0(r)) + +a(\dot{y}_1(r) + \dot{y}_0(r)) - g(r, \dot{y}_0(r)) \\ &= 0. \\ Also \dot{p}^{(k)}(0) &= \dot{y}_1^{(k)}(0) - \dot{y}_0^{k}(0) = 0, \\ \dot{p}(1) &= \dot{y}_1(1) - \dot{y}_0(1), \\ &\leq \int_0^1 \dot{y}_1(s)\, d\eta(s) - \int_0^1 \dot{y}_0(s)\, d\eta(s), \\ &= \int_0^1 [\dot{y}_1(s) - \dot{y}_0(s)]\, d\eta(s) = \int_0^1 \dot{p}(s)\, d\eta(s). \end{aligned}$$

Then by Lemma 4.1, $\dot{p}(r) \leq 0 \Rightarrow \dot{y}_1(r) \leq \dot{y}_0(r)$, $\forall r \in (0, 1)$.
Now to prove $\dot{x}_1(r) \leq \dot{y}_1(r)$ $\forall r \in (0, 1)$. Set $\dot{p}(r) = \dot{y}_1(r) - \dot{x}_1(r)$. Then by $[B3]$ and

definition of $\dot{x}_1(r)$, $\dot{y}_1(r)$, we get

$$\begin{aligned}
-D^{\nu}_{0+}\dot{p}(r) &= -D^{\nu}_{0+}[\dot{y}_1(r)-\dot{x}_1(r)],\\
&= -D^{\nu}_{0+}\dot{y}_1(r)-[-D^{\nu}_{0+}\dot{x}_1(r)],\\
&= g(r,\dot{y}_0(r))-a[\dot{y}_1(r)-\dot{y}_0(r)]-[g(r,\dot{x}_0(r))-a[\dot{x}_1(r)-\dot{x}_0(r)]],\\
&= [g(r,\dot{y}_0(r))-g(r,\dot{x}_0(r))]-a[\dot{y}_1(r)-\dot{y}_0(r)]+a[\dot{x}_1(r)-\dot{x}_0(r)],\\
&\geq -a(\dot{y}_0(r)-\dot{x}_0(r))-a[\dot{y}_1(r)-\dot{y}_0(r)]+a[\dot{x}_1(r)-\dot{x}_0(r)],\\
&= -a(\dot{y}_1(r)-\dot{x}_1(r)) = -a\dot{p}(r).
\end{aligned}$$

$$Also \dot{p}^{(k)}(0) = \dot{y}_1^{(k)}(0)-\dot{x}_1^{(k)}(0)=0,$$

$$\begin{aligned}
\dot{p}(1) = \dot{y}_1(1)-\dot{x}_1(1) &= \int_0^1 \dot{y}_1(s)\,d\eta(s)-\int_0^1 \dot{y}_1(s)\,d\eta(s),\\
&= \int_0^1 \dot{p}(s)\,d\eta(s).
\end{aligned}$$

Then by Lemma 4.1, $\dot{p}(r)\geq 0 \Rightarrow \dot{x}_1(r)\leq \dot{y}_1(r)\ \forall r\in(0,1)$. Now by mathematical induction method, it is easy to verify that

$$\dot{x}_0(r)\leq \dot{x}_1(r)\leq \dot{x}_2(r)\leq \ldots \leq \dot{x}_m(r)\leq \dot{y}_m(r)\leq \ldots \dot{y}_1(r)\leq \dot{y}_0(r).$$

Thus the sequences $\{\dot{x}_m\}$, $\{\dot{y}_m\}$ are uniformly bounded and monotonically non-decreasing and non-increasing in $\mathscr{C}$. Hence the point-wise limit exist and are given by $\underset{m\to\infty}{lim}\ \dot{x}_m(r)=\dot{x}(r)$, $\underset{m\to\infty}{lim}\ \dot{y}_m(r)=\dot{y}(r)$ on $\mathscr{C}$. Next we claim that $\dot{x}(r)$ and $\dot{y}(r)$ are the extremal solutions of BVP (1.1). Let $z(r)$ be any solution of BVP (1.1) different from $\dot{x}(r)$ and $\dot{y}(r)$ in Ω^*. So there exist some i such that $\dot{x}_i(r)\leq z(r)\leq \dot{y}_i(r),\ r\in(0,1)$. Set $\dot{p}_1(r)=z(r)-\dot{x}_{i+1}(r)$. So that, by assumption $[B3]$, we obtain

$$\begin{aligned}
-D^{\nu}\dot{p}_1(r) &= -D^{\nu}z(r)-(-D^{\nu}\dot{x}_{i+1}),\\
&= g(r,z(r))-[g(r,\dot{x}_i(r))-a(\dot{x}_{i+1}(r)-\dot{x}_i(r))],\\
&= [g(r,z(r))-g(r,\dot{x}_i(r))]+a(\dot{x}_{i+1}(r)-\dot{x}_i(r)),\\
&\geq -a(z(r)-\dot{x}_i(r))+a(\dot{x}_{i+1}(r)-\dot{x}_i(r)),\\
&= -a[z(r)-\dot{x}_i(r)-\dot{x}_{i+1}(r)+\dot{x}_i(r)],\\
&= -a(z(r)-\dot{x}_{i+1}(r)) = -a\dot{p}_1(r),\\
\dot{p}_1^{(k)}(0) &= 0, \quad \dot{p}_1(1)=\int_0^1 \dot{p}_1(s)\,d\eta(s).
\end{aligned}$$

Then by Lemma 4.1, $\dot{p}_1(r)\geq 0$ implying that $\dot{x}_{i+1}(r)\leq z(r)$ for all i. Similarly set $\dot{p}_2(r)=\dot{y}_{i+1}(r)-z(r)$ and using $[B3]$ we obtain

$$\begin{aligned}
-D^{\nu}\dot{p}_2(r) &= -D^{\nu}\dot{y}_{i+1}(r)-(-D^{\nu}z(r)),\\
&= [g(r,\dot{y}_i(r))-a(\dot{y}_{i+1}(r)-\dot{y}_i(r))]-g(r,z(r)),\\
&= [g(r,\dot{y}_i(r))-g(r,z(r))]-a(\dot{y}_{i+1}(r)-\dot{y}_i(r)),\\
&\geq -a(\dot{y}_i(r)-z(r))+a(\dot{y}_{i+1}(r)-\dot{y}_i(r)),\\
&= -a[\dot{y}_i(r)-z(r)-\dot{y}_{i+1}(r)+\dot{y}_i(r)],\\
&= -a(\dot{y}_{i+1}(r)-z(r)) = -a\dot{p}_2(r),\\
\dot{p}_2^{(k)}(0) &= 0, \quad \dot{p}_2(1)=\int_0^1 \dot{p}_2(s)\,d\eta(s).
\end{aligned}$$

Then by Lemma 4.1, $\dot{p}_2(r) \geq 0$ implying that $z(r) \leq \dot{y}_{i+1}(r)$ for all i. Hence $\dot{x}_{i+1}(r) \leq z(r) \leq \dot{y}_{i+1}(r)$, $r \in (0,1)$. Since $\dot{x}_0(r) \leq z(r) \leq \dot{y}_0(r)$ on $\mathscr{C}$. Hence by induction method, it follows that $\dot{x}_i(r) \leq z(r) \leq \dot{y}_i(r)$ for all i. Taking limit as $i \to \infty$, it follows that $\dot{x}(r) \leq z(r) \leq \dot{y}(r)$ on $\mathscr{C}$. Thus the functions $\dot{x}(r)$, $\dot{y}(r)$ are the extremal solutions of the BVP (1.1). The proof is complete.

Next we prove uniqueness of solutions of the BVP (1.1).

Theorem 4.3. *Assume that,*

(i) $[B_1]$, $[B_2]$, $[B_3]$ *holds,*
(ii) *there exists $a > 0$ such that the function g satisfies the condition*

$$g(r, v) - g(r, v^*) \leq a(v - v^*) \tag{4.2}$$

for $\dot{x}_0 \leq v \leq v^* \leq \dot{y}_0, r \in (0,1)$.

Then the BVP (1.1) has a unique solution in Ω^*.

Proof:- We know $\dot{x}(r) \leq \dot{y}(r)$ on $\mathscr{C}$. It is sufficient to prove that $\dot{x}(r) \geq \dot{y}(r)$. Consider $\dot{p}(r) = \dot{y}(r) - \dot{x}(r)$. Then we have

$$\begin{aligned} -D^\nu \dot{p}(r) &= -D^\nu \dot{y}(r) - (-D^\nu \dot{x}(r)), \\ &= g(r, \dot{y}(r)) - g(r, \dot{x}(r)), \\ &\leq -a(\dot{y}(r) - \dot{x}(r)) = -a\dot{p}(r) \end{aligned}$$

and

$$\dot{p}^{(k)}(0) = 0, \quad \dot{p}(1) = \int_0^1 \dot{p}(s)\, d\eta(s).$$

By Lemma 4.1, $\dot{p}(r) \leq 0$ implying that $\dot{y}(r) \leq \dot{x}(r)$. Hence $\dot{x}(r) = \dot{y}(r)$ is the unique solution of BVP (1.1).

5. Conclusion

By implementing Banach contraction mapping theorem, it is shown that the mapping T has a unique fixed point in $\mathscr{C}$. Monotone iterative sequences $\{\dot{x}_m\}$ and $\{\dot{y}_m\}$ converging uniformly to $\dot{x}(r)$ and $\dot{y}(r)$ as $m \to \infty$ respectively are constructed. Monotone technique developed is applied to prove that $\dot{x}(r)$, $\dot{y}(r)$ are minimal and maximal solutions of problem (1.1) in Ω^*. Uniqueness of solutions of the nonlinear problem (1.1) with integral boundary conditions is also obtained.

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