Integral Equations & Linear Integral Equations

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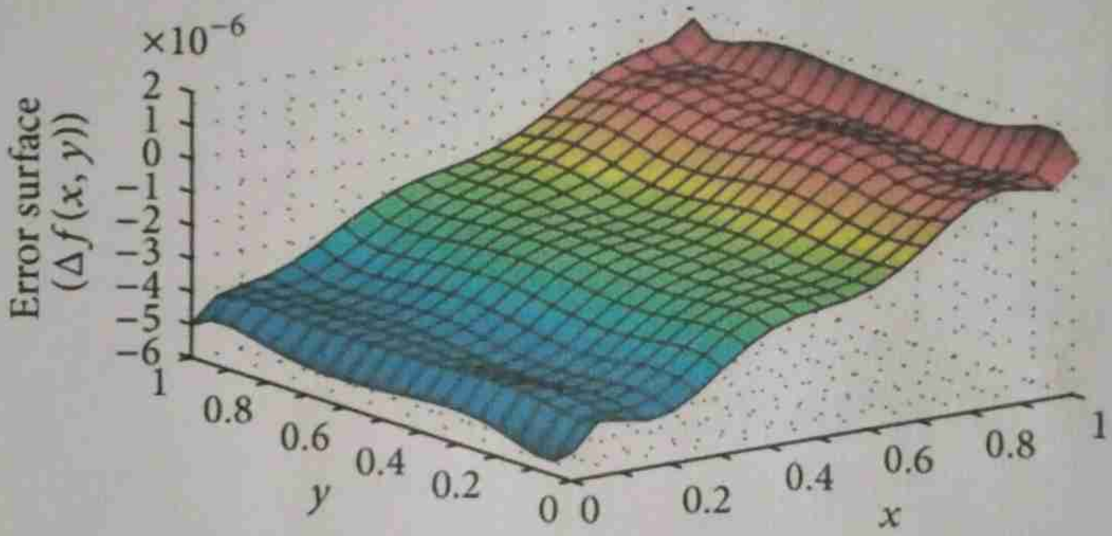

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Introduction

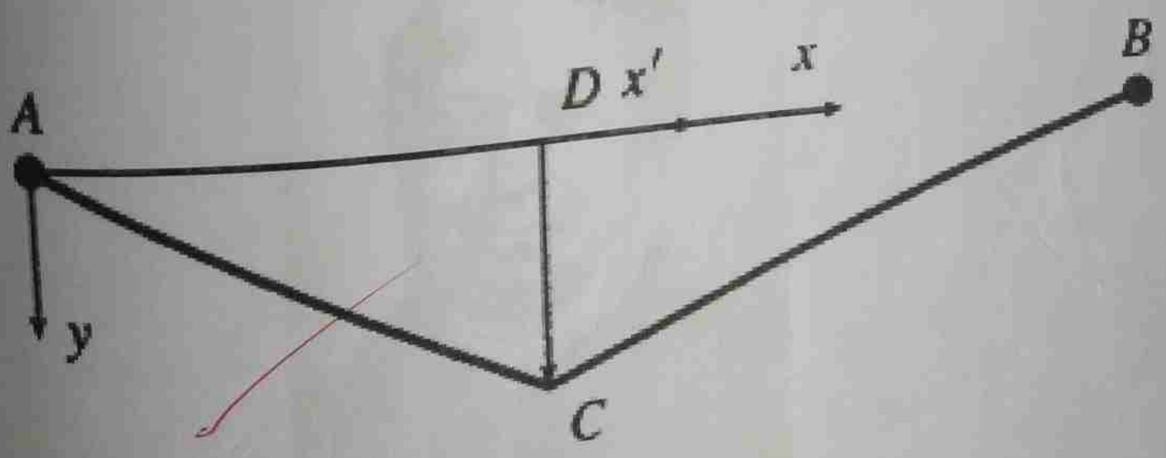
An integral equation is an equation in which an unknown function appears under one or more integration signs.

Any integral calculus statement like

$y = \int_a^b \phi(x)dx$ can be considered as an integral equation.

A general type of integral equation, g(x) y(x) = f(x) + $\lambda \int_a^b K(x,t)$ y(t) dt is called linear integral equation as only linear operations are performed in the equation.

The one, which is not linear, is called 'Non-linear integral equation'.



History

In 1823 Abel proposed a generalization of the tautochrone problem whose solution involved the solution of an integral equation which has more recently been designated as an integral equation of the first kind, and in 1837 Liouville showed that the determination of a particular solution of a linear differential equation of the second order could be effected by solving an integral equation of a different type, called the integral equation of the second kind. The ripple of mathematical interest which had its origin in these investigations increased at first but slowly. Recently, however, stimulated by the researches of Volterra, Fredholm, and Hilbert in the period between 1896 and the present time, which seemed at first only a ripple has grown into a formidable wave.

JOSEPH LIOUVILLE

Definition

An equation containing the unknown function under the integral sign. Integral equations can be divided into two main classes: linear and non-linear integral equations.

Linear integral equations have the form

$$A(x)\varphi(x)+\int_D K(x,s)\,\varphi(s)ds=d(x),\quad x\in D$$

where A, K, f are given functions, A being called the coefficient, K the kernel and f the free term (or right-hand side) of the integral equation, D is a bounded or unbounded domain in a one- or higher-dimensional Euclidean space, x,s are points of this space, ds is the volume element, and φ is the unknown function.

Types of Linear Integral Equations

There are four basic types of integral equations. They are as follows:

$$\begin{aligned}
\int_a^x K(x,y)\varphi(y)\,dy &= f(x) \\
\varphi(x) - \int_a^x K(x,y)\varphi(y)\,dy &= f(x) \\
\int_a^b K(x,y)\varphi(y)\,dy &= f(x) \\
\varphi(x) - \int_a^b K(x,y)\varphi(y)\,dy &= f(x)
\end{aligned}$$

All four involve the unknown function $\phi(x)$ in an integral with a kernel $K(x, y)$ and all have an input function $f(x)$. In all four the integration ranges from some fixed lower limit. In the **Volterra equations**, the upper limit of integration is the variable x, while in the **Fredholm equations**, the upper limit of integration is a fixed constant. The so-called equations **of the first kind** only involve the unknown function ϕ inside the integral. The equations **of the second kind** also involve ϕ outside the integral.

So the four equations above are

- Volterra equation of the first kind
- Volterra equation of the second kind
- Fredholm equation of the first kind
- Fredholm equation of the second kind

Apart from these, there are references to Volterra or Fredholm equations of the **third kind**. These are an extension of the second kind, where a function *A*(*x*) multiples the ϕ outside the integral. Equations of the second kind are the most important since the first and third kinds can often be reduced to the second kind.

	Volterra	Fredholm
1st	$\int_a^x K\,\varphi$	$\int_a^b K\,\varphi$
2nd	$\varphi - \int_a^x K\,\varphi$	$\varphi - \int_a^b K\,\varphi$

Applications

The study of integral equations has a significant role in both pure and applied field of mathematics. The development of science has led to the formation of many physical laws, which when restated in mathematical form, often appear as differential equations. Engineering problems can be mathematically described by differential equations and thus differential equations play very important roles in the solution of practical problems. For example, Newton's law, stating that the rate of change of the momentum of a particle is equal to the force acting on it, can be translated into mathematical language as a differential equation. Similarly, problems arising in electric circuits, chemical kinetics and transfer of heat in a medium can all be represented mathematically as differential equations. These differential equations can be transformed to the equivalent integral equations of Volterra and Fredholm types. There are many physical problems that are governed by integral equations and these equations can be easily transformed to the differential equation. Applications of some special integral equations are tested below.

1 Abel's integral equation: This equation arises in the problem of finding the path of a particle which is constrained to move under gravity in a vertical plane.

2 Vandrey's equation: This equation occurs in fluid dynamics while calculating the pressure distribution on the surface of the body of revolution moving in a fluid.

3 Seismic Response of Dams: In order to analyze the safety and stability of an earth dam during an earthquake we need to know the response of the dam to earthquake ground motion so that the inertia forces that will be generated in the dam by the earthquake can be derived. Once the inertia forces are known, the safety and the stability of the structure can be determined. The inertia forces generated during an earthquake will depend on the geometry of the

dam, the material properties, earthquake time history. The reality of the problem is that an earth dam is a three dimensional structure. The material properties are non- linear inelastic and the earthquake time history is a time varying phenomenon. The problem is therefore very complex and a proper solution requires the use of a finite element program, which can deal with nonlinear inelastic material properties. The earthquake is the travelling wave phenomenon, which arrives at the base of the dam through the foundation rock. Since the foundation is not rigid, part of the energy, which vibrates the dam is lost through the foundation causing radiation damping. The analytical solution to such a problem is not at all possible. However, it is often necessary to have approximate solutions that can be used to understand the behaviour of the dam during earthquakes. In order to make the problem amenable to analytical solutions, some approximations are made to create a mathematical model. Such a model in this case is known as the shear beam model (SB) of earth dams. Here we formulate a differential equation using the ideas from physics such as shear modulus, strain, inertia, acceleration, momentum and so on. And we convert this differential equation to an integral equation to get solution in an easier way.

5 Transverse Oscillations of a Homogenous Elastic Bar: Consider a homogeneous elastic bar with linear mass density d. Its axis coincides with the segment $(0, l)$ of the s axis when the bar is in its state of rest. It is clamped at the end $s = 0$, free at the end $s = l$, and is forced to perform simple harmonic oscillations with period $2\pi/\omega$, ω is the angular frequency.