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* Introduction *

The theory of integral equations was first introduced by J. Fourier. Fourier (1768-1830) is the initiator of the theory of integral equations. The term integral equation was first suggested by Du-Bois Reymond in 1888. The pioneering systematic investigations goes back to late 19^{th} and early 20^{th} century works of Volterra, Fredholm & Hilbert.

The study here mainly points towards linear integral equations especially on Fredholm and Volterra type of integral equations. One of the main relevance of this study arise from the idea of conversion of Differential equations into integral equations and vice versa.

Main Topic

Integral Equation and Linear Integral Equations

Integral Equation

* **Definition** –

An integral equation is an equation in which an unknown function appears under one or more integral signs.

For Example – For $a \le s \le b$; $a \le t \le b$
The equations

$$f(s) = \int_a^b K(s,t)\, g(t)\, dt , \quad \text{---} \; (1)$$

$$g(s) = f(s) + \int_a^b K(s,t)\, g(t)\, dt ; \quad \text{---} \; (2)$$

$$g(s) = \int_a^b K(s,t)\, [g(t)]^2\, dt , \quad \text{---} \; (3)$$

where, the function $g(s)$ is the unknown function and all the other functions are known, are integral equations. These functions of the complex-valued functions of the real variables s and t.

* Classification of Integral Equations -

An integral equation can be classified in to different categories as we have seen in ordinary and partial differential equations.

We can classify integral equations as,

1) Linear or Nonlinear,
2) Homogeneous or Nonhomogeneous.

The most frequently used integral equations fall under two major classes, namely

1) Fredholm integral Equations
2) Volterra integral Equations.

* Linear Integral Equations -

An integral equation is called Linear if and only iff linear operations are performed in it upon the unknown function.

For a linear integral equation, the unknown function $u(x)$ appearing under integral sign is given in the functional form $F(u(x))$ such that the power of $u(x)$ is unity.

The most general type of linear integral equation is of the form,

$$h(x)\, g(x) = f(x) + \lambda \int_a k(x,t) g(t)\, dt$$

where, the upper limit may be either variable or fixed.

The functions $h(x)$, $f(x)$, $k(x,t)$ are known functions while $g(x)$ is to be determined, λ is the non-zero real or complex parameter. The function $k(x,t)$ is called the kernel.

* Example - 1)

$$u(x) = f(x) + \int_0^x K(x,t)\, u(t)\, dt$$

If the unknown function $u(x)$ appearing under integral sign is given in the functional form $F(u(x))$ such that, the power of $u(x)$ is no longer unity, then such integral equations can be classified as Nonlinear integral equations.

* Example - 2)

$$u(x) = f(x) + \int_0^x K(x,t)\, u^2(t)\, dt$$

Throughout this study we mainly focus on Linear Integral Equations.

1) **Fredholm integral equations** -

The most standard form of fredholm linear integral equations is given by the form.

$$\phi(x)\, u(x) = f(x) + \lambda \int_a^b k(x,t)\, u(t)\, dt \quad - (1)$$

where the limits of integration a and b are constants and the unknown function $u(x)$ appears linearly under the integral sign. $k(x,t)$ is the kernel of the eq^n.

a) If $\phi(x) = 1$, then [1] becomes

$$u(x) = f(x) + \lambda \int_a^b k(x,t)\, u(t)\, dt \quad - (2)$$

and this eq^n is called Frendholm integral equation of second kind.

b) If $\phi(x) = 0$, ~~then~~, (1) yields

$$f(x) + \lambda \int_a^b k(x,t)\, u(t)\, dt = 0 \quad - (3)$$

which is called fredholm integral equation of the first kind.

c) The homogeneous fredholm integral eq^n of the second kind is a special case of (b) above In this case $f(x) = 0$,

$$u(x) = \lambda \int_a^b k(x,t)\, u(t)\, dt \quad - (4)$$

2) Volterra integral equations. –

The most standard form of volterra linear integral equations is of the form.

$$\phi(x)\, u(x) = f(x) + \lambda \int_a^x K(x,t)\, u(t)\, dt \quad - (1)$$

where the limits of integration are functions of x and the unknown $u(x)$ appears linearly under the integral sign.

If the function $\phi(x) = 1$, then – (1) becomes

$$u(x) = f(x) + \lambda \int_a^x K(x,t)\, u(t)\, dt \quad - (2)$$

and this equation is known as the volterra integral equation of the second kind.

If $\phi(x) = 0$, then (1) becomes

$$f(x) + \lambda \int_a^x K(x,t)\, u(t)\, dt = 0 \quad - (3)$$

which is the volterra integral equation of first kind.

3) **Singular and Non-Singular integral equations –**

A singular integral equation is defined as an integral with the infinite limits when the kernel $K(x,t)$ of the integral becomes unbounded at a certain point in the interval.

Example –
$$u(x) = f(x) + \lambda \int_{-\infty}^{+\infty} u(t)\, dt$$

If the kernel $K(x,t)$ is bounded and continuous then the integral equation is said to be non-singular.

Remark –

If we set $f(x)=0$, in Volterra and fredhlom integral equations then the resulting equation is called a homogeneous integral equation, otherwise it is called non-homogeneous.

* Relation Between Differential and Integral Equations –

Integral and differential equations have a fundamental importance in functional analysis and practice problems. But in many cases the resolution of differential equations with constant coefficients is easy but the resolution of these equations with variable coefficients is practically difficult or impossible in more part of the cases.

Here. we present an analytical method which transforms a differential equation into integral equation..

Also, it presents methods to convert an integral equation into a differential equation.

* Transformation of Differential Equations into Integral Equations -

A boundary value problem of the form $y'' + y' f_1(x) + y f_2(x) = f_3(x)$ with $y(a) = y_0$ and $y(b) = y_1$ can be transformed in to a fredholm eq^n. An initial value problem of the form

$$y'' + y' f_1(x) + y f_2(x) = f_3(x) \quad \text{with } y(a) = y_0$$

& $y(b) = y_1$ can be transformed in to a volterra eq^n.

* **Example** - Construct the linear eq integral eq^n corresponding to the differential eq^n - $y'' + xy = 1$ with initial conditions $y(0) = y'(0) = 0$

Solution - Given. $y'' = 1 - xy$

Integrate w.r.t. x from 0 to x

$$\int_0^x y''(x)\,dx = \int_0^x (1 - xy)\,dx$$

$$\Rightarrow [y'(x)]_0^x = [x]_0^x - \int_0^x xy(x)\,dx$$

Integrat w.r.t x from 0 to x

$$\int_0^x [y'(x) - y'(0)]\,dx = \int_0^x (x - 0)\,dx - \int_0^x \left[\int_0^x xy(x)\,dx\right] dx$$

$$\Rightarrow [y(x)]_0^x = \left[\frac{x^2}{2}\right]_0^x - \frac{1}{(2-1)!}\int_0^x (x-t)^{2-1} t y(t)\,dt$$

$$\Rightarrow y(x) - y(0) = \frac{x^2}{2} - \int_0^x (x-t)\, t\, y(t)\, dt$$

$$\Rightarrow y(x) \cdot = \frac{x^2}{2} - \int_0^x (x-t) + y(t)\, dt$$

which is the required linear volterra integral equation.

* <u>Transformation of Integral equations into Differential equations</u> –

If the upper limit is x and kernel doesn't depend on x, then volterra equation is equivalent to an ordinary differential equation.

$$\phi(x) = f(x) + \lambda \int_0^x K(x,t)\, \phi(t)\, dt$$

$$\Rightarrow \phi'(x) = f'(x) + \lambda K(x)\, \phi(x)$$

If the upper limit is x and kernel depends on x then it is a possible to reduce the integral equation to an ordinary differential equation by differentiating several times w.r.t x.

* Conclusion *

This study briefly explains the concepts associated with an integral equation, especially linear integral equation. It mainly points towards linear integral equations such as Fredholm and Volterra integral equations.

There are a number of methods to solve the two types of integral equations. We have discussed a few among them in this study. One obvious reason for using integral equations rather than differential equations is that all of the conditions specifying the initial value problem or boundary value problem for a differential equation can often be condensed into a single integral equation.