SHRI KRISHNA MAHAVIDAYALAYA, GUNJOTI.

DEPARTMENT OF MATHEMATCS

NAME	**AAMIR SOHAIL ABDUL SALAM**
PRN NO.	**2017015200195924**
CLASS	**M.SC MATH. 4 SEM (2021-2022)**
PROJECT PAPER	**LINEAR INTREGAL EQUTION**
PAPER NO.	**MAT -511**
TITLE OF THE PROJECT	**INTEGRAL EQUATION AND LINEAR INTEGRAL EQUTIONS**

HOD

1 History of integral equation 1888-1903

J. Fourier (1768-1830) is the initiator of the theory of integral equations. A term integral equation first suggested by Du Bois-Reymond in 1888. Du Bois-Reymond define an integral equation is understood an equation in which the unknown function occurs under one or more signs of definite integration. Late eighteenth and early ninetieth century Laplace, Fourier, Poission, Liouville and Able studies some special type of integral equation. The pioneering systematic investigations goes back to late nineteenth and early twentieth century work of volterra, Fredholm and Hilbert. In 1887, Volterra published a series of famous papers in which he singled out the notion of a functional and pioneered in the development of a theory of functional s in theory of linear integral equation of special type. Fredholm presented the fundamentals of the Fredholm integral equation theory in a paper published in 1903 in the Acta Mathematica. This paper became famous almost overnight and soon took its rightful place among the gems of modern mathematics. Hilbert followed Fredholm's famous paper with a series of papers in the Nachrichten of the GUttingen Academy.

1.1 Functional

A functional is a generalization of the notion of a function of a finite number of numerical variables, defined by Aristotle D. Michal.

1.2 Integral equation

Integral equation is an equation in which the unknown, say a function of a numerical variable, occurs under an integral. That means a functional equation involving the unknown function under one or more integrals. For example,

$$y(x) = f(x) + 5\int_0^1 e^{t+x} y(t)dt$$

1.3 Volterra Integral equation

Second kind of Linear Volterra integral equation defined by

$$y(x) = f(x) + \lambda \int_{x_0}^{x} K(t,x)y(t)dt \tag{1}$$

where $f(x), K(t,x)$ are known functions and $y(x)$ is the unknown function and λ is a numerical parameter.
Second kind of non - linear Volterra integral equation defined by

$$y(x) = f(x) + \lambda \int_{x_0}^{x} K_0(t,x,y(t))dt \tag{2}$$

where $K_0(t,x,y(t)) \neq K(t,x)y(t)$. First kind of Linear Volterra integral equation defined by

$$f(x) + \lambda \int_{x_0}^{x} K(t,x)y(t)dt = 0 \tag{3}$$

First kind of non - linear Volterra integral equation defined by

$$f(x) + \lambda \int_{x_0}^{x} K_0(t,x,y(t))dt = 0 \tag{4}$$

1.4 Fredholm Integral equation

Second kind of Linear Fredholm integral equation defined by

$$y(x) = f(x) + \lambda \int_{x_0}^{x_1} K(t,x)y(t)dt \tag{5}$$

where $f(x), K(t,x)$ are known functions and $y(x)$ is the unknown function and λ is a numerical parameter.
Second kind of non - linear Fredholm integral equation defined by

$$y(x) = f(x) + \lambda \int_{x_0}^{x_1} K_0(t,x,y(t))dt \tag{6}$$

where $K_0(t,x,y(t)) \neq K(t,x)y(t)$.
First kind of Linear Fredholm integral equation defined by

$$f(x) + \lambda \int_{x_0}^{x_1} K(t,x)y(t)dt = 0 \tag{7}$$

First kind of non - linear Fredholm integral equation defined by

$$f(x) + \lambda \int_{x_0}^{x_1} K_0(t, x, y(t))dt = 0 \tag{8}$$

Remarks If $f(x) = 0$ then above defined Volterra and Fredholm Integral equations are called homogeneous type otherwise non homogeneous. Main difference between Volterra and Fredholm Integral equations are range of integration in the integral equation.

1.5 Volterra Fredholm Integral equation

We consider an integral equation

$$y(x) = C_1 + \frac{C_2 - C_1}{x_1 - x_0}(x - x_0) - \frac{x - x_0}{x_1 - x_0}\int_{x_0}^{x_1}(x_1 - t)\psi(t)dt + \int_{x_0}^{x}(x - t)\psi(t)dt \tag{9}$$

In this equation, there are two types of integrals. First one has finite range and other has variable range. There are two way to think about this integral equation according to domain.

Case 1. If $(x, t) \in [x_0, x_1]$ and $x_0, x_1 \in R$ then above equation can be written in the form.

$$y(x) = C_1 + \frac{C_2 - C_1}{x_1 - x_0}(x - x_0) + \frac{1}{x_1 - x_0}\int_{x_0}^{x_1} H(x, t)\psi(t)dt \tag{10}$$

where

$$H(x, t) = \begin{cases} -(x - x_0)(x_1 - t) + (x_1 - x_0)(x - t) & , \quad x \le t \\ -(x - x_0)(x_1 - t) & , \quad x \ge t. \end{cases}$$

This equation look like a Fredholm integral equation.

Case 2. $t \in [x_0, x_1]$ and $x \in R$ then, we can not covert as described in case 1. So such type of integral equation called Non homogeneous linear Volterra - Fredholm integral equation.

2 Solution of integral equation

A solution of integral equation is a function that satisfies the original integral equation.

Verify that the given functions are solutions of the corresponding integral equations and check it kinds.

- $y(x) = x - \frac{x^3}{6}$, $y(x) = x - \int_0^x \sin(x - t)y(t)dt$

- $y(x) = 1 - x,\ x = \int\limits_0^x e^{x-t} y(t)dt$

- $y(x) = 1,\ y(x) = e^x - x - \int\limits_0^1 x\left(e^{xt} - 1\right) y(t)dt$

- $y(x) = 1,\ y(x) = 1 + \gamma(x) - \int\limits_0^\infty e^{-t} t^{x-1} y(t)dt$

- $y(x) = 1,\ B(x,q) = \int\limits_0^1 t^{x-1}(1-t)^{q-1}\, y(t)dt,\ Rep > 0,\ Req > 0.$

2.1 Reduction of first kind Volterra integral equation in to second kind Volterra integral equation

Suppose

$$\psi(x) = \int\limits_{u1(x)}^{u2(x)} F(x,t)dt \tag{11}$$

then

$$\psi^{(1)}(x) = \int\limits_{u1(x)}^{u2(x)} \frac{\partial F(x,t)}{\partial x} dt + F(u2(x),t)\frac{\partial u2(x)}{\partial x} - F(u1(x),t)\frac{\partial u1(x)}{\partial x} \tag{12}$$

We consider a linear first kind Volterra integral equation

$$f(x) + \lambda \int\limits_{x_0}^{x} K(t,x) y(t)dt = 0 \tag{13}$$

Differentiating with respect to x, we get

$$f^{(1)}(x) + \lambda \int\limits_{x_0}^{x} \frac{\partial K(x,t)}{\partial x} y(t)dt + K(x,x)y(x) = 0 \tag{14}$$

2.2 Relation between initial value problem of differential equation and Volterra integral equation

The solution of the linear differential equation

$$\sum_{k=0}^{N} a_k(x) y^{(k)}(x) = F(x) \tag{15}$$

with continuous coefficients $a_k(x), k = 0, 1, ..., N$, given the initial conditions

$$y^{(k)}(x_0) = C_k, k = 0, 1, ..., N-1. \tag{16}$$

may be reduced to a solution of some Volterra integral equation of the second kind.
For example, we take $N = 2$ the above initial vale problem reduces second order differential equation. Suppose

$$y^{(2)}(x) = \psi(x). \tag{17}$$

Integrating two times and using initial conditions and Leibnitz formula for n times integration,

$$\int_{x_0}^{x} \frac{(x-t)^{n-1}}{n-1|}\psi(t)dt = \int_{x_0}^{x}\int_{x_0}^{x}...\int_{x_0}^{x}\psi(t)dtdt...dt$$

we get

$$y^{(1)}(x) = C_1 + \int_{x_0}^{x}\psi(t)dt \tag{18}$$

and

$$y(x) = C_0 + (x - x_0)C_1 + \int_{x_0}^{x}(x-t)\psi(t)dt \tag{19}$$

Substituting $y^{(2)}(x)$, $y^{(1)}(x)$, $y(x)$ from Equs (11)-(13) in the original differential equation, we get

$$\psi(x) = f(x) + \int_{x_0}^{x} K(x,t)\psi(t)dt \tag{20}$$

where

$$f(x) = F(x)$$

and

$$K(x,t)_{=}C$$

The theory of partial differential equations is another important source of integral equations. For instance, the first-order partial differential equation

$$u_t = g(t,x,u) - \sum_{k=1}^{n} f_i(t,x)u_{x_i} \tag{21}$$

with an initial condition of the form

$$u(0,x) = h(x),\ x \in D \subset R^n \tag{22}$$

can be reduced to an integral equation of Volterra type, if appropriate conditions are satisfied. First, let us notice that the right-hand side of Eq. (15) represents the derivative of the function $u(t,x)$ along the trajectories of the differential system Eq. (15). That mean we can substitute $x = x(t) - a$ in Eqs. (15) - (16). After this Eqs. (15) - (16) become a initial value problem of ordinary differential equation.

2.3 Exercises

Reduce initial value problem of differential equation in to particular Volterra integral equation

- $y^{(2)}(x) + xy^{(1)}(x) + y(x) = 0,\ y(0) = 1,\ y^{(1)}(0) = 0$
- $y^{(2)}(x) + y(x) = \cos(x),\ y(0) = 0,\ y^{(1)}(0) = 0$
- $y^{(2)}(x) - \sin(x)y^{(1)}(x) + e^{x}y(x) = x,\ y(0) = 1,\ y^{()}(0) = -1$
- $y^{(3)}(x) - 2xy(x) = 0,\ y(0) = 1/2,\ y^{(1)}(0) = y^{(2)}(0) = 0$
- Initial value problem of heat equation
- Initial value problem of wave equation
- Initial value problem of Laplace equation

The basic problems arising in connection with a given functional equation are the following:

- The existence of solutions;
- The uniqueness problem — that is, what conditions should be added in order to determine a single solution;
- The construction of the solution.

3 Construction of the solution of Volterra integral equation

In previous section we showed an initial value problem of differential equation is equivalent to Volterra integral equation. First idea to solve such type of problems, we convert this problem in an initial value problem of differential equation. After this the solution of initial value problem of differential equation is the required solution. There are several way to construct the solution, we are going to illustrate some of then as follows.

4 Picard's method of successive approximation

Picard's constructed the solution of Volterra integral equation given by

$$y_0(x) = f(x) \tag{23}$$

$$y_n(x) = f(x) + \lambda \int_{x_0}^{x} K_0(t, x, y_{n-1}(t))dt \tag{24}$$

For linear Volterra integral equation, we construct a sequence of a function as follow

$$y_1(x) = f(x) + \lambda \int_{x_0}^{x} K(t,x) f(t) dt$$

$$y_2(x) = f(x) + \lambda \int_{x_0}^{x} K_1(t,x) f(t) dt + \lambda^2 \int_{x_0}^{x} K_2(t,x) f(t) dt$$

$$\vdots$$

$$y_n(x) = f(x) + \int_{x_0}^{x} \sum_{k=1}^{n} \lambda^k K_k(t,x) f(t) dt \tag{25}$$

where

$$K_n(t,x) = \int_{x_0}^{x} K(t,u) K_{n-1}(u,x) du, \; n \geq 2$$

$$y_n(x) = f(x) + \int_{x_0}^{x} R_n(t,x,\lambda) f(t) dt \tag{26}$$

where

$$R_n(t,x,\lambda) = \sum_{k=1}^{n} \lambda^k K_k(t,x)$$

Question arises that when the sequence of function $y_n(x)$ is uniformly convergent. We only need to find the condition for series $\sum_{k=1}^{\infty} \lambda^k K_k(t,x)$ (resolvent kernel) is uniformly convergent.

We have the condition

$$|R_n(t,x)| \leq \frac{\lambda^n M^n (x-t)^{n-1}}{n-1|}$$

with assumption that $K(t,s) \leq M, \; x_0 \leq t \leq x \leq a, \; a > 0$. These estimates prove that $R(t,x,\lambda) = \sum_{k=1}^{\infty} \lambda^k K_k(t,x)$ is uniformly convergent on $x_0 \leq t \leq x \leq a, \; a > 0$.

Taking limit both sides as n approaches to infinity, we get

$$y(x) = f(x) + \int_{x_0}^{x} R(t,x,\lambda) f(t) dt \tag{27}$$

This formula gives the solution and it shows that $R(t,x,\lambda)$ serves to represent the solution for any continuous $f(x)$ Because of this region we call it resolvent kernel.

5 Existence and uniqueness solution for Volterra integral equation

Theorem 5.1 *Let us consider Eq. (2) under the following assumptions. (i) f(x) is continuous on the domain* $[x_0, a]$, $a > 0$. *(ii)* $K_0(t, x, y)$ *is a continuous function on D, where* $D = \{(t, x, y) : x_0 \leq t \leq x \leq x_0 + a,\ f(x) - b \leq y \leq f(x) + b,\ b > 0\}$. *(iii)* $K_0(t, x, y)$ *satisfies Lipschitz condition on D. Then, there exists a unique continuous solution* $y(x)$ *of Eq. (2) defined for* $x_0 \leq x \leq x_0 + \delta$, *where* $\delta = min\left\{a, \frac{b}{M}\right\}$ *with* $M = \sup |K_0(t, x, y)|$ *on D.*

The method of successive approximations consists in constructing the sequence of continuous functions

$$y_n(x) = f(x) + \lambda \int_{x_0}^{x} K_0(t, x, y_{n-1}(t))dt \tag{28}$$

We consider

$$\begin{aligned} |y_n(x) - y_{n-1}(x)| &\leq \lambda \int_{x_0}^{x} |K_0(t, x, y_{n-1}(t)) - K_0(t, x, y_{n-2}(t))|dt \\ &\leq \lambda L \int_{x_0}^{x} |y_{n-1}(t) - y_{n-2}(t)|dt \\ &\leq \lambda M L^{n-1} \frac{(x - x_0)^n}{n|} \end{aligned} \tag{29}$$

$$|z(x) - y_n(x)| \leq \lambda M L^n \frac{(x - x_0)^n}{n|} \tag{30}$$

5.1 Exercise

Check that existence and uniqueness of the solution of first kind linear Volterra integral equation with $f(x) = 0$, where kernels are given by

$$k(x, t) = \begin{cases} te^{1/x^2 - 1} & , \quad 0 \leq t \leq e^{1-1/x^2} \\ x & , \quad xe^{1-1/x^2} \leq t \leq x \\ 0, & \quad t \geq x \end{cases}$$

$$K(x, t) = t^{x-t}$$

$$K(x, t) = \frac{xt^2 2}{\pi(x^6 + t^2)}$$

6 Relation between boundary value problem of linear ordinary differential equation and Fredholm integral equation

We consider boundary value problem of linear Nth order ordinary differential equation

$$\sum_{k=0}^{N} a_k(x)y^{(k)}(x) = F(x) \tag{31}$$

$$\sum_{k=0}^{N-1} b_{j,k}y^{(k)}(x_j) = C_j,\ j = 1, 2 ..., N. \tag{32}$$

For convenient, we consider boundary value problem of linear second order ordinary differential equation

$$y^{(2)}(x) + a_0(x)y^{(1)}(x) + a_1(x)y(x) = r(x) \tag{33}$$

$$y(x_0) = C_1 \tag{34}$$

$$y(x_1) = C_2 \tag{35}$$

Suppose

$$y^{(2)}(x) = \psi(x). \tag{36}$$

Integrating two times from 0 to x and using boundary condition, we get

$$y^{(1)}(x) = \frac{C_2 - C_1}{x_1 - x_0}(x - x_0) - \frac{x - x_0}{x_1 - x_0}\int_{x_0}^{x_1}(x_1 - t)\psi(t)dt + \int_{x_0}^{x}\psi(t)dt \tag{37}$$

$$y^{(1)}(x) = \frac{C_2 - C_1}{x_1 - x_0}(x-x_0) - \frac{1}{x_1 - x_0}\int_{x_0}^{x}[(x-x_0)(x_1-t)+(x_0-x_1)]\psi(t)dt - \frac{x - x_0}{x_1 - x_0}\int_{x}^{x_1}(x_1-t)\psi(t)dt \tag{38}$$

$$y^{(1)}(x) = \frac{C_2 - C_1}{x_1 - x_0}(x - x_0) + \frac{1}{x_1 - x_0}\int_{x_0}^{x_1} G(x,t)\psi(t)dt \tag{39}$$

where

$$G(x,t) = \begin{cases} -(x - x_0)(x_1 - t) + (x_1 - x_0) & , \quad x \le t \\ -(x - x_0)(x_1 - t) & ; \quad x \ge t. \end{cases} \tag{40}$$

$$y(x) = C_1 + \frac{C_2 - C_1}{x_1 - x_0}(x - x_0) - \frac{x - x_0}{x_1 - x_0}\int_{x_0}^{x_1}(x_1 - t)\psi(t)dt + \int_{x_0}^{x}(x-t)\psi(t)dt \tag{41}$$

$$y(x) = C_1 + \frac{C_2 - C_1}{x_1 - x_0}(x - x_0) + \frac{1}{x_1 - x_0}\int_{x_0}^{x_1} H(x,t)\psi(t)dt \qquad (42)$$

where

$$H(x,t) = \begin{cases} -(x - x_0)(x_1 - t) + (x_1 - x_0)(x - t) & , \quad x \leq t \\ -(x - x_0)(x_1 - t) & , \quad x \geq t. \end{cases} \qquad (43)$$

6.1 Exercises

Reduce initial value problem of differential equation in to particular Volterra integral equation

- $y^{(2)}(x) + xy^{(1)}(x) + y(x) = 0, \; y(0) = 1, \; y^{(1)}(1) = 0$
- $y^{(2)}(x) + y(x) = \cos(x), \; y(0) = 0, \; y^{(1)}(1) = 0$
- $y^{(2)}(x) - \sin(x)y^{(1)}(x) + e^{x}y(x) = x, \; y(0) = 1, \; y^{()}(1) = -1$
- $y^{(3)}(x) - 2xy(x) = 0, \; y(0) = 1/2, \; y^{(1)}(0) = y^{(2)}(1) = 0$
- Boundary value problem of heat equation
- Boundary value problem of wave equation
- Boundary value problem of Laplace equation

7 Existence and uniqueness solution for Fredholm integral equation

Theorem 7.1 *Let us consider Eq. (3) under the following assumptions. (i) f(x) is a continuous complex - valued function on* $[x_0, x_1]$. *(ii)* $K(t, x)$ *is a continuous function on* $[x_0, x_1]\times[x_0, x_1]$. *(iii)* $|\lambda|M\,(x_1 - x_0) < 1$, *where* $M = \sup|K(t, x)|$ *. Then, there exists a unique continuous solution* $y(x)$ *of Eq. (3).*

8 Method of solution for Fredholm integral equation

References

[1] Aristotle D. Michal. (1950). Integral equation and functional. Mathematical Association of America. 24, 2, 83-95.

[2] C. Corduneanu. (1977). Principles of Differential and Integral Equations. Chelsea Publishing Company The Bronx, New York.