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- Linear Algebra

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MAT - 535

Rank of linear transformation

* Linear Transformation:

The central Objective of linear algebra is the analysis of the shear transformation is a this sort. First we define the concept of a linear function or transformation

Defⁿ-
Let V and W be real vector space (their dimensions can be different) and let T be a function with domain V and range in W (written $T: V \to W$)
We say T is a linear transformation if

a) for all $x, y \in V$, $T(x+y) = T(x) + T(y)$ (T is addtive)

b) For all $x \in V$, $r \in \mathbb{R}$, $T(rx) = rT(x)$ (T is homogeneous)

If V and W are complex vector spaces the defⁿ. is the same except in
b) $r \in \mathbb{C}$
If $V = W$, then T can be called a linear operater.

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E.g. let $V = W = E'$ Define $T(x) = mx$, wh[e]
m is a fixed real number. Show that T
linear transformation.

Sol^n

We must show that T is additive and
geneous for the additivity.
We let x and y be in E' and Calculate

$$T(x+y) = m(x+y) = mx + my$$
$$T(x) + T(y) = mx + my$$

Since $T(x+y) = T(x) + T(y)$, we know tha[t]
is additive. Also T is homogeneous si[nce]

$$T(rx) = m(rx) = (mr)x = r(mx) = rT$$

$\therefore$ Thus T is a linear transformation

E.g. Let $V = W = E'$. For $x \in V$, define $F(x)$
$= mx + b$, where m and b are real numbe[rs]
and $b \neq 0$. Show that F is not a linear [trans]
-formation

Sol^n

First we check additivity, noting $F(\cdot) =$

$$F(x+y) = m(x+y) + b$$
$$= mx + my + b$$

However

$$F(x) + F(y) = (mx + b) + (my + b)$$
$$= mx + my + 2b$$

since $b \neq 0$, $2b \neq 0$. so $F(x+y) \neq F(x) + F(y)$ for all all $x, y \in V$, and F is not linear

THEOREM

Let V and W be vector space. If $T: V \to W$ is a linear transformation then $T(\theta_V) = \theta_W$. (The subscripts emphasize the vector space that the zero vector comes from)

Proof -

Since $\theta_V + \theta_V = \theta_V$,

$$\underbrace{T(\theta_V)}_{\text{In } W} = T(\theta_V + \theta_V) \underset{\text{By additivity}}{=} \underbrace{T(\theta_V) + T(\theta_V)}_{\text{In } W}$$

and so

$$T(\theta_V) = T(\theta_V) + T(\theta_V)$$

By uniqueness of θ_W in W the only way the last equation can hold is if

$$T(\theta_V) = \theta_W$$

This thm can sometimes be used to show transformations are non-linear. A logical consequence of the thm is

If $T(\theta_V) \neq \theta_W$ then T is not linear

Eg Show that $T: E^2 \to E^2$; defined by

$$T((x_1, x_2)) = (x_1 + x_2, x_1 - x_2 + 1)$$

is not linear

Eg Let $T: E^2 \to E^1$ be defined by

$$T((x_1, x_2)) = x_1^2 + x_2^2$$

Show that T is not linear even though $T(\theta) = \theta$.

Sol^n = We have $T(\theta) = T(0,0)) = 0^2 + 0^2 = 0$ which is the zero of E^1 this allows no Conclusion. the definition of linearity must be used

To check additivity we Calculate

$$T(x+y) = T((x_1, x_2) + (y_1, y_2))$$

$$= T((x_1 + y_1, x_2 + y_2))$$

$$= (x_1 + y_2)^2 + (x_2 + y_2)^2$$

$$= x_1^2 + 2x_1y_1 + y_1^2 + x_2^2 + 2x_2y_2 + y_2^2$$

and

$$T(x) + T(y) = T((x_1, x_2)) + T(y_1, y_2))$$

$$= x_1^2 + x_2^2 + y_1^2 + y_2^2.$$

Since $T(x+y) \neq T(x) + T(y)$. We know that T is not linear. In most cases, to determine linearity or nonlinearity of a transformation. We use the definition

Show that the following transformation are linear.

a) $T: E^3 \to E^3$ defined by

$$T(x_1, x_2, x_3)) = (x_1 + x_2, x_2 + x_3, x_3 + x_1)$$

b) $T: E^3 \to E^3$ defined by

$$T(x_1 x_2, x_3) = v \times (x_1\ x_2\ x_3)$$

where v is a fixed vector in E^3

c) $T: E^3 \to E'$ defined by

$$T(x_1, x_2, x_3)) = ax_1 + bx_2 + cx_3$$

where a, b and c are fixed real numbers

d) $T: M_{22} \to M_{22}$ defined by

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

e) $T: P_1 \to P_2$ defined by

$$T(ax+b) = \frac{ax^2}{2} + bx$$

F) $T: \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$T((z_1, z_2)) = (z_1 + z_2, z_1 - 2z_2$$

solution- Parts (a) through (e) are left t
the problems.

(F) $T((z_1, z_2) + (u_1, u_2)) = T(z_1 + u_1, z_2 + u$

$= (z_1 + u_1 + z_2 + u_2, z_1 + u_1 - 2z_2$

$= (z_1 + z_2, z_1 - 2z_2) + (u_1 + u_2,$

$= T(z_1, z_2) + T(u_1, u_2)$

$T(c(z_1, z_2)) = T(cz_1, cz_2)$

$= (cz_1 + cz_2, cz_1 - 2cz_2)$

$= c(z_1 + z_2, z_1 - 2z_2)$

$= cT(z_1, z_2)$

Thus T is linear

Show that $T: C_{22} \to C_{22}$ defined by $T(A) = \bar{A}$ is not linear

Sol^n - We know that $T(cA) = \overline{cA} = \bar{c}\bar{A} = \bar{c}T(A)$
$\neq cT(A) = \bar{A}$ is not linear
unless $c \in \mathbb{R}$.
but c can have that a non-zero imaginary part, so T is not linear (However T is called conjugate linear because $T(cA) = \bar{c}T(A)$ and $T(A+B) = T(A) + T(B)$

Let $V = M_{n1}$ and $W = M_{m1}$. Let M be an $m \times n$ real matrix Define $T: V \to W$ by

$$T(x) = Mx$$

T is linear because by matrix algebra

$$T(x+y) = M(x+y) = Mx + My$$

$$T(cx) = M(cx) = c(Mx)$$

let $V = C_{n1}$ and $W = C_{m1}$ and let Z be an $m \times n$ matrix from C_{mn}. Define $T: V \to W$ by $T(x) = Zx$. Then T is linear because by matrix algebra

$$Z(x+y) = Zx + Zy$$
$$Z(cx) = c(Zx)$$

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Some Special linear transformation m
be noted for future use
The zero transformation T_θ from V
is defined as

$$T(x) = \theta_W \quad \text{for all } x \text{ in } V$$

The identity transformation I from V to
is defined as

$$I(x) = x \quad \text{for all } x \text{ in } V$$

The contraction transformation $T_\circ$ from V
is

$$T_\circ(x) = \alpha x \quad 0 < \alpha < 1, \text{ for all}$$

The dilation transformation T_β from V
is

$$T_\beta x = \beta x \quad 1 < \beta \text{ for all } x \in$$

Verification that these are linear tran
formation is left to the problems.

Although several eg linear transformat
have now been given.
We have not yet begun to analyze
linear transformation
In algebra, analysis of function was a
with graphs of the function.

In our present situation we must usually be satisfied without the types of graphs we drew in algebra

Find $\ker T$, where $T : E^3 \to E^2$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, x_2 - x_3)$

– Since $\ker T = \{x \mid T(x) = \theta\}$ we must solve $T(x_1, x_2, x_3)\} = (0, 0)$.

$$(x_1 + x_2, x_2 - x_3) = (0, 0)$$

The resulting equation are

$$x_1 + x_2 = 0$$
$$x_2 - x_3 = 0$$

Which have solution $(-k, k, k)$ Therefore

$$\ker T = \{v \in E^3 \mid v = k(-1, 1, 1)\}$$
$$= \operatorname{Span}\{(-1, 1, 1)\}$$

Let V and W be vector space, and let $T : V \to W$ be a linear transformation. The set $\ker T$ is a Subspace of V

Proof – The kernel of T is non-empty because $T(\theta) = \theta$. We need to show that $\ker T$ is closed under addition and Scalar multiplication

Recall that $x \in \ker T$ if and only
let x and y be in ker T
and c be a number
By the linearity of T.

$$T(x+y) = T(x) + T(y)$$
$$= \theta + \theta$$
$$= \theta$$
$$T(cx) = cT(x)$$
$$= c\theta$$
$$= \theta$$

So $x+y \in \ker T$ and cx ker T, thus T is a subspace of V

Since ker T is a subspace of V, it dimension. The dimension of ker T is the nulity of T.

Thus for the linear transformation the nullity is 1

We write this

$$\eta(T) = 1$$

e.g. Calculate $\eta(T)$ for the linear transforn $T: E^3 \to E^2$ defined by

$$T((a,b,c)) = (a+2b+c \quad -a+3b+$$

Find a basis for ker T.

We must find the set of all vectors (a,b,c) in E^3 that $T(a,b,c) = (0,0)$
That is the equation

$$\begin{pmatrix} a+2b+c \\ -a+3b+c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

must be solved. The solution is

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -k \\ -2k \\ 5k \end{pmatrix}$$

and $\ker T = \text{span}\{(-1,-2,5)\}$. Therefore dim $(\ker T) = 1$, so $n(T) = 1$. A basis is $\{(-1,-2,5)\}$.

To continue the analysis of linear transformation, we consider the <u>range</u> of T.
In algebra finding the range of a function f is important in graphing $y = f(x)$

eg $y = x^2 - 2x - 3$ has range $\{y \mid -4 \leq y \leq \infty\}$
the soln of $x^2 - 2x - 3 = 0$ are $x = 3, x = -1$

(That is, the kernel of f is $\{-1, 3\}$. All this information is shown.
The range of a linear transformation cannot always be used to obtain a graph of T.
but it is quite useful in other ways

Def^n - Let $T: V \to W$ be a linear transformation. The range of T is the set of all possible such that $y = T(x)$. for same x in V.
The range of T is the set of all possible in V in W. such that $y = T(x)$ for some x in V.
The range of T is written range T. The range of T is a subspace of W (see the

Th^m - If $T: V \to W$ is a linear transformation and $\dim V = n$, then

$$R(T) + n(T) = n$$

Before proving this th^m, we consider an of its use.

e.g Find the nullity of the linear transformation

Sol^n We have $T: E^3 \to E^3$ and found $R(T)$
since $\dim V = \dim E^3 = 3$
leads to

$$2 + n(T) = 3$$

Therefore $n(T) = 1$

Proof - Since ker T and range of T are vector space, $R(T)$ and $n(T)$ are defined. consider three case $n(T) = 0$, $n(T) = n$ and
$1 \leq n(T) \leq n-1$

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Case I.

$n(T) = 0$, Suppose $R(T) = k < n$. That is suppose that $n(T) + R(T) < n$

We will obtain a contradiction

Since $R(T) = k$, any set of more than k vectors in range T is linearly dependent

Let $\{v_1 \cdots v_n\}$ be a basis for V

Since $k < n$, $\{T(v_1) \cdots T(v_n)\}$

must be linearly dependent and so there exist