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cayley- Hamilton theorem.

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Cayley - Hamilton Theorem -

Cayley-Hamilton theorem, characteristic Polynomial.

let $A = [a_{ij}]$ be an n-square matrix. The matrix $M = A - tI_n$ Where I_n is the n-square identity matrix and t is an indeterminate, may be obtained by subtracting t down the diagonal of A. The negative of M is the matrix $tI_n - A$ and it's determinat.

$$\Delta(t) = \det(tI_n - A) = (-1)^n \det(A - tI_n).$$

Which is a Polynomial in t of degree n and is called characteristic Polynomial of A.

(Cayley - Hamilton) Every matrix A is a root of it's characteristic Polynomial of A.

Every square matrix satisfies it's own characteristic equation.

or

$$|A - \lambda I| = (-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + --- + a_n)$$

be the characteristic Polynomial, then the equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + --- + a_nI = 0$$

is satisfied by $x = A$.

i.e. $A^n + a_1A^{n-1} + --- a_2A^{n-2} + a_nI = 0.$

→ let A be a square matrix of order n

It's characteristic equation

$$|A - \lambda I| = 0.$$

$$P_0 + P_1\lambda + P_2\lambda^2 + ---- + P_{n-1}\lambda^{n-1} + P_n\lambda^n = 0$$

Where P_0, P_1 ----- P_n are scaler.

R.T.P =

$$P_0 I + P_1 A + P_2 A^2 + \cdots + P_n A^n = 0$$

$$A \ \mathrm{adj}\ A = |A| I$$

$$(A - \lambda I)\ \mathrm{adj}\ (A - \lambda I) = |A - \lambda I|\ I \quad \text{——— (a)}$$

$$\mathrm{adj}(A - \lambda I) = B_0 + B_1 \lambda + B_2 \lambda^2 + \cdots + B_{n-1} \lambda^{n-1}$$

——— (b)

order of $A = n$

Put the value of (b) in (a)

$$\Rightarrow (A - \lambda I)(B_0 + B_1 \lambda + B_2 \lambda^2 + \cdots + B_{n-1}\lambda^{n-1}) = (P_0 + P_1 \lambda + \cdots + P_n \lambda^n) I$$

$$\Rightarrow AB_0 + AB_1 + AB_2 \lambda^2 + \cdots + AB_{n-1}\lambda^{n-1}$$

$$\lambda I B_0 - \lambda^2 I B_1 \cdots \cdots - \lambda^n I B_{n-1}$$

$$= P_0 I + P_2 I + \cancel{\cdots} + P_{n-1}\lambda^{n-1} I + P_n \lambda^n I$$

vimal

L.H.S =

$$AB_0 + (AB_1 - B_0\lambda) + (AB_2 - B_1)\lambda^2 + \cdots + (AB_{n-1} - B_{n-2})\lambda^{n-1}I + \cancel{\cdots} + -\lambda^n B_{n-1}.$$

R.H.S =

$$P_0 I + P_1\lambda I + P_2\lambda^2 I + \cdots + p^n\lambda^n I$$

$$AB_0 = P_0 I$$
$$AB_1 - B_0 = P_1 I$$
$$AB_2 - B_1 = P_2 I$$
$$\vdots$$
$$AB_{n-1} - B_{n-2} = P_{n-1} I$$
$$B_{n-1} = P_n I$$

$$P_0 I + P_1 A + P_2 A^2 + \cdots + P_{n-1}A^{n-1} + P_n A^n$$
$$= AB_0 + A^2B_1 - AB_0 + A^3B_2 - A^2B_1 \cdots A^nB_{n-1} - A^{n-1}B_{n-2} - A^nB_{n-1}$$

Hence proved.

Example -

verify cayley - Hamilton theorem -

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$\longrightarrow$

The characteristic equation of A is

$|A - \lambda I| = 0$

implies

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

operating $R_1 \rightarrow R_1 + R_2$, We get

$$\begin{vmatrix} 1-\lambda & 1-\lambda & 0 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

operating $C_2 \rightarrow C_2 - C_1$, We get -

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ -1 & 3-\lambda & -1 \\ 1 & -2 & 2-\lambda \end{vmatrix} = 0$$

on expanding through first row,
We get -

$$(1-\lambda)\left[(3-\lambda)(2-\lambda)-2\right]=0$$

implies $\lambda^3+6\lambda^2-9\lambda+4=0$ ----- (i)

This is charasteristic equation.

Now, for cayley-Hamilton theorem, matrix A must satisfy Eq (i). So, on Putting $\lambda=A$ We get,

$$A^3-6A^2+9A-4I=0 \quad \text{-----} \; (ii)$$

Now,

$$A^2=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}$$

vimal

$$A^3 = A^2 \times A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

on Putting the values of A, A^2, A^3 and I in RHS of Eq^n (ii), We get.

$$A^3 - 6A^2 + 9A - 4I$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$+ 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= 0$$

Hence, the cayley-Hamilton theorem is verified.