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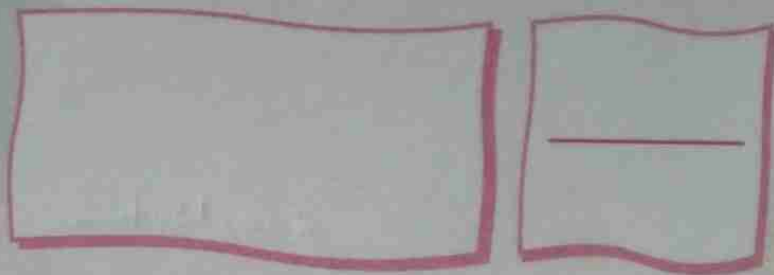
Project Name - Linear Algebra

Title of the Project -
Cayley-Hamilton theorem.

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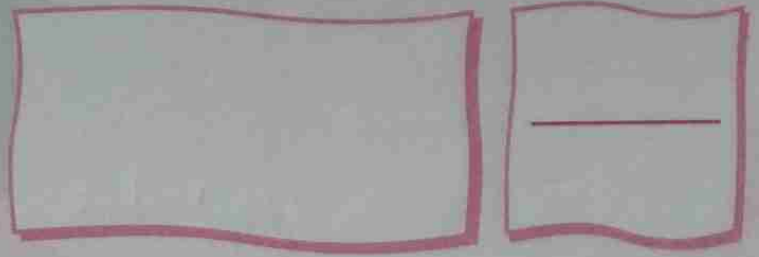


Cayley - Hamilton Theorem - Cayley-Hamilton theorem, characteristic Polynomial.

Let $A = [a_{ij}]$ be an n -square matrix. The matrix $M = A - tI_n$ where I_n is the n -square identity matrix and t is an indeterminate, may be obtained by subtracting t down the diagonal of A . The negative of M is the matrix $tI_n - A$ and its determinant.

$$\Delta(t) = \det(tI_n - A) = (-1)^n \det(A - tI_n).$$

Which is a Polynomial in t of degree n and is called characteristic Polynomial of A .



(Cayley - Hamilton) Every matrix A is a root of its characteristic Polynomial of A .

Every square matrix satisfies its own characteristic equation.

or

$$|A - \lambda I| = (-1)^n (\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots + a_n)$$

be the characteristic Polynomial, then the equation

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n I = 0$$

is satisfied by $x = A$.

i.e. $A^n + a_1 A^{n-1} + \dots + a_2 A^{n-2} + a_n I = 0$.

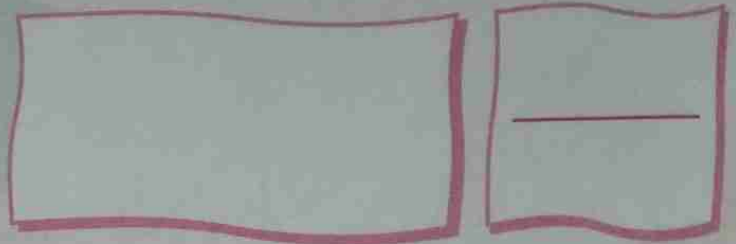
→ let A be a square matrix of order n

Its characteristic equation

$$|A - \lambda I| = 0$$

$$P_0 + P_1 \lambda + P_2 \lambda^2 + \dots + P_{n-1} \lambda^{n-1} + P_n \lambda^n = 0$$

Where P_0, P_1, \dots, P_n are scalar.



R.T.P =

$$P_0 I + P_1 A + P_2 A^2 + \dots + P_n A^n = 0$$

$$A \operatorname{adj} A = |A| I$$

$$(A - \lambda I) \operatorname{adj}(A - \lambda I) = |A - \lambda I| I \quad \text{--- (a)}$$

$$\operatorname{adj}(A - \lambda I) = B_0 + B_1 \lambda + B_2 \lambda^2 + \dots + B_{n-1} \lambda^{n-1}$$

order of $A = n$ (b)

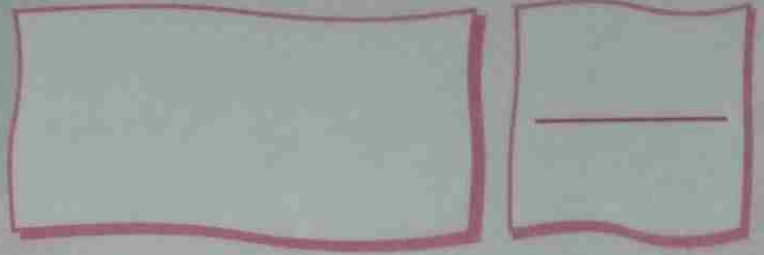
Put the value of (b) in (a)

$$\Rightarrow (A - \lambda I) (B_0 + B_1 \lambda + B_2 \lambda^2 + \dots + B_{n-1} \lambda^{n-1}) = (P_0 + P_1 \lambda + \dots + P_n \lambda^n) I$$

$$\Rightarrow AB_0 + AB_1 \lambda + AB_2 \lambda^2 + \dots + AB_{n-1} \lambda^{n-1}$$

$$- \lambda I B_0 - \lambda^2 I B_1 - \dots - \lambda^n I B_{n-1}$$

$$= P_0 I + P_2 I + \dots + P_{n-1} \lambda^{n-1} I + P_n \lambda^n I$$



L.H.S =

$$AB_0 + (AB_1 - B_0d) + (AB_2 - B_1d^2) + \dots + (AB_{n-1} - B_{n-2}d^{n-1}) + \dots - d^n B_{n-1}$$

R.H.S =

$$P_0 I + P_1 d I + P_2 d^2 I + \dots + P_n d^n I$$

$$AB_0 = P_0 I$$

$$AB_1 - B_0 = P_1 I$$

$$AB_2 - B_1 = P_2 I$$

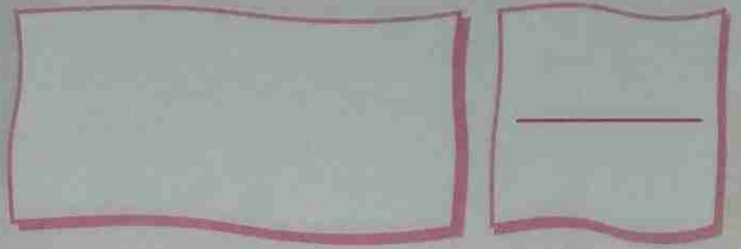
⋮

$$AB_{n-1} - B_{n-2} = P_{n-1} I$$

$$B_{n-1} = P_n I$$

$$\begin{aligned} & P_0 I + P_1 A + P_2 A^2 + \dots + P_{n-1} A^{n-1} + P_n A^n \\ &= AB_0 + A^2 B_1 - AB_0 + A^3 B_2 - A^2 B_1 - \dots \\ & \quad A^n B_{n-1} - A^{n-1} B_{n-2} - A^n B_{n-1} \end{aligned}$$

Hence proved.



Example -

verify Cayley-Hamilton theorem -

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

→

The characteristic equation of A is

$$|A - \lambda I| = 0$$

implies

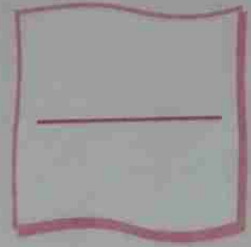
$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

operating $R_1 \rightarrow R_1 + R_2$, We get

$$\begin{vmatrix} 1-\lambda & 1-\lambda & 0 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

operating $C_2 \rightarrow C_2 - C_1$, We get -

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ -1 & 3-\lambda & -1 \\ 1 & -2 & 2-\lambda \end{vmatrix} = 0$$



on expanding through first row ,
We get -
 $(1-\lambda) [(3-\lambda)(2-\lambda) - 2] = 0$

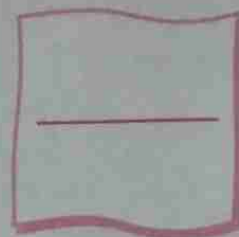
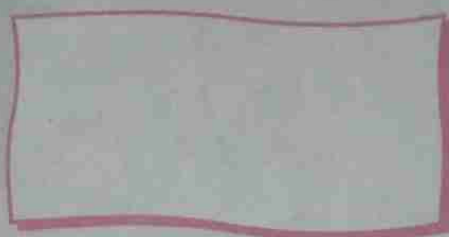
implies $\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$ ----- (i)
This is characteristic equation.

Now, for Cayley-Hamilton theorem, matrix
A must satisfy Eq (i). So,
on putting $\lambda = A$ We get,
 $A^3 - 6A^2 + 9A - 4I = 0$ ----- (ii)

Now,

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$



$$A^3 = A^2 \times A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

on Putting the values of A, A^2, A^3 and I in RHS of Eqⁿ (ii), We get.

$$A^3 - 6A^2 + 9A - 4I$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$+ 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

~~= 0~~
Hence, the Cayley-Hamilton theorem is verified.