Page No.: Date :

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INDEX

Introduction

he calculus of variations is an
tesive subject and there are many
ne reference which present a
etail development of the subject
he purposse of this addendum is do
rovid a brief background in the
theory behind Lagrange's Equations
ortunately comple understanding of this
heory is not absolutely neccessary
o use Lagrage's equation but a basic
nderstanding of variational principles
an greatly increase your mechanical
nodeling skill

xtremum of an Integral-
The Euler-Lagrange Eqn

Given the Integral of a functional of e from

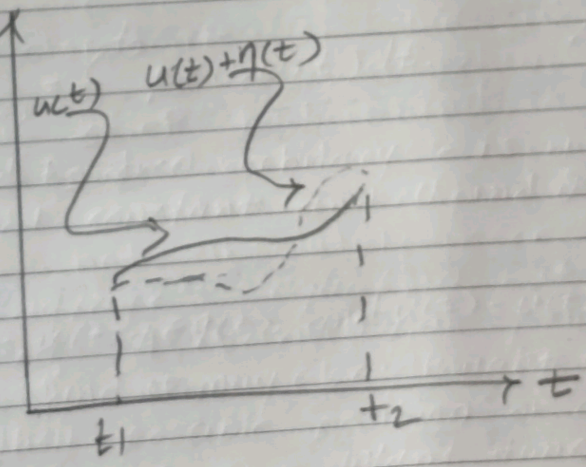
$$I_{(\epsilon)} = \int_{t_1}^{t_2} F(U, \dot{U}, t)\, dt. \quad \text{—— (I)}$$

where t_1 and t_2 are arbitary ϵ is a small positive real constant and U and $\dot{U}$ are given by

$$U(t) = u(t) + \epsilon\eta(t), \text{ and } \dot{U}(t) = \dot{u}(t) + \epsilon\dot{\eta}(t) \quad \text{(II)}$$

The function U and u may be thought of as describing the possible position of a dynamical system bet^n the two intant time t_1 and t_2 where it is an extremum and $U(t)$ is $u(t)$ plus a variation $\epsilon\eta(t)$ the function $U(t)$ does not by defined render (1) stationary because we shall assume $\eta(t)$ is independent of $u(t)$ and we will assume that a unique function renders (1) an extremm the reason for these assumption will become clear below The impotant point so far is that have not much made any restrictive statement about $I(\epsilon)$ other than it is an integral

Now that we have stage more or less up lets see what rule the functional $f(U, \dot{U}, t)$ must obey to render (1) extreme we have defination that the function $u(t)$ renders I stationary hence we know this occurs when $\bar{u}(t) = u(t)$ or $\epsilon = 0$ this situation is depicted in figure



Thus assuming that t_1 and t_2 are not functions of ϵ we set first derivatives of $I(\epsilon)$ equal to zero

$$\left.\frac{dI(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \int_{t_1}^{t_2} \frac{df}{d\epsilon}(U, \dot{U}, t)\, dt$$

However

$$\frac{df}{d\epsilon}(U, \dot{U}, t) = \frac{\partial f}{\partial U}\frac{\partial U}{\partial \epsilon} + \frac{\partial f}{\partial \dot{U}}\frac{\partial \dot{U}}{\partial \epsilon}$$

So substituting Equation (4) into Equation (3) and setting $\epsilon = 0$ we have

$$\frac{dI(\epsilon)}{d\epsilon}\Big|_{\epsilon=0} \int_{t_1}^{t_2} \left(\frac{\partial F}{\partial u} + \frac{\partial F}{\partial \dot{u}}\dot{\eta}\right) dt = 0 \quad \text{--- (5)}$$

Integration of Equation (5) by parts yields

$$\frac{dI(\epsilon)}{d\epsilon}\Big|_{\epsilon=0} = \int_{t_1}^{t_2} \eta(t)\left(\frac{\partial F}{\partial u} - \frac{d}{dt}\frac{\partial F}{\partial \dot{u}}\right) dt + \frac{\partial F}{\partial \dot{u}}\eta(t)\Big|_{t_1}^{t_2} = 0 \quad \text{--- (6)}$$

The last term in equation (6) vanishes because of the stipulation $\eta(t_1) = \eta(t_2) = 0$ which leaves

$$\frac{dI(\epsilon)}{d\epsilon}\Big|_{\epsilon=0} = \int_{t_1}^{t_2} \eta(t)\left(\frac{\partial F}{\partial u} - \frac{d}{dt}\frac{\partial F}{\partial \dot{u}}\right) dt = 0 \quad \text{--- (7)}$$

$$\frac{dI(\epsilon)}{d\epsilon}\Big|_{\epsilon=0} = \frac{\partial F}{\partial u} - \frac{d}{du}\frac{\partial F}{\partial \dot{u}} = 0 \quad \text{--- (8)}$$

Equation (5) is often written

$$\delta I = \frac{dI(\epsilon)}{d\epsilon}\Big|_{\epsilon=0} = \epsilon \int_{t_1}^{t_2} \left(\frac{\partial F}{\partial u} + \frac{\partial F}{\partial \dot{u}}\right) dt$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial \dot{u}}\delta \dot{u}\right) dt = \quad \text{--- (9)}$$

$\delta I = \eta$ is the variation of u and (2)

$$\frac{d}{dt}\delta u = \frac{d}{dt}(\epsilon\eta) = \epsilon\dot{\eta} = \delta\frac{du}{dt} = \delta\dot{u} \quad (10)$$

using Equation (10) and integrating Equation (9) by part we obtain

$$\delta I = \int_{t_1}^{t_2}\left(\frac{\partial F}{\partial u} - \frac{d}{du}\frac{\partial F}{\partial \dot{u}}\right)\delta u \, dt \quad (11)$$

with the stipulation as before that $\delta u(t_1)$
$= \delta u(t_2) = 0$

Hamilton's principle

Hamilton's principles is perhaps the most important result in the calculus of variation we derived the Euler-Lagrange equation for a single variable u but we will now shift our attention fo system N particles of mass mi Hence we may obtain N equation of the from

$$m_i \ddot{r}_i = F_i \quad \text{(12)}$$

where the bold font indicates a vector quan and F_i denotes the total force on the i^{th} particle D' Alem principles may be state by rewriting Equation (12) as

$$m_i \ddot{r}_i - F_i = 0 \quad \text{(13)}$$

$$\sum_{i=1}^{N} (m_i \ddot{r}_i - F_i)\delta r = 0 \quad \text{(14)}$$

we note that the sum of the virtual wo done by the applied forces over the virtual displacement is given by

$$\delta W = \sum_{i=1}^{N} f_i \, \delta r \quad \text{(15)}$$

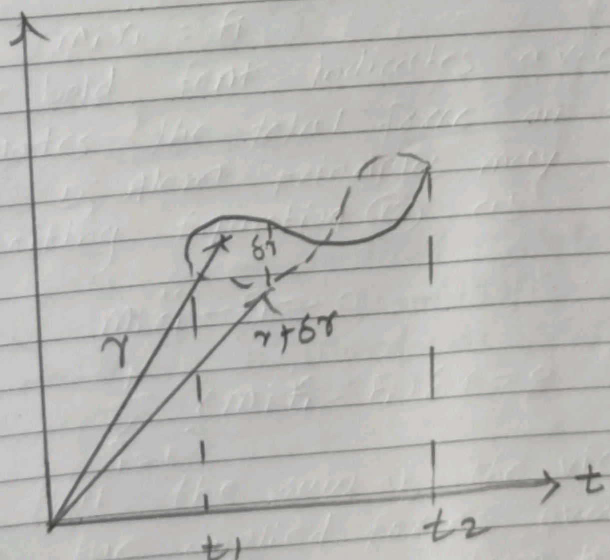
Next we note that

$$\sum_{i=1}^{N} m_i \ddot{r}_i \, \delta r_i = \sum_{i=1}^{N} m_i \left[\frac{d}{dt}(\dot{r}_i \cdot \delta r_i) - \delta\left(\frac{1}{2}\dot{r}_i \cdot \dot{r}_i\right)\right]$$

$$= \sum_{i=1}^{N} m_i \frac{d}{dt}(\dot{r}_i \, \delta r_i) - \delta T \quad \text{(}$$

$$\delta T + \delta W = \sum_{i=1}^{N} m_i \frac{d}{dt}(\dot{r}_i \, \delta r_i) \quad \text{—— (17)}$$

In manner similar to that shown figure and inview of equation (18) the possible dynamical path of each paricle may be represented as shown fig where the varied dynamical path may be thought to occur atemporally



possible dynamical path for a particale betn two arbitary intants in time

$$\int_{t_1}^{t_2} (\delta T + \delta W)\,dt = \sum_{i=1}^{N} m_i (\dot{r}_i \cdot \delta r_i)\Big|_{t_1}^{t_2} = 0 \qquad (18)$$

If δW can be expressed as the variation of the potential energy δV eqn (18)

$$\int_{t_1}^{t_2} (\delta T - \delta V)\,dt = 0 \qquad (19)$$

Introducting the Lagrange Function

$$L = T - V \quad \text{Eq } (19)$$

$$\delta \int_{t_1}^{t_2} L\,dt = 0 \qquad (20)$$

Eq. (20) is the mathematical statement of Hamilton's principles Hamilton's principle may be defined in word as follow

Defination →

The actual path a body takes in configuration space renders the values of the definite integral $I = \int_{t_1}^{t_2} L\,dt$ stationary with respect to all arbitrary variations of the path betn two instant t_1 and t_2 provided that the path variations vanish at t_1 and

In all physical system the stationary value of I will be minimum.

Application of Lagrange multipliers to compute Equilibrium Reaction force

Theorem → The total work done by the forces acting on a body in equilibrium during a reversible virtual displacement cosistent with the system constraints is zero

$$\delta W = -\delta W \quad \text{--- (1)}$$

At equilibrium we have that

$$\delta W = -\delta V = 0 \quad \text{--- (2)}$$

eqn (1) implies the well know fact that the potential energy is minimum at a stable equilibrium

$$\delta V = \frac{\delta V}{\delta x_1}\delta x_1 + \frac{dv}{dx_2} + \dots + \frac{dv}{dx_n} dx_n$$

We consider the same system to be subject to a constraint of the form.

$$\phi(x_1, x_2, \dots, x_n) = 0 \quad \text{--- (3)}$$

Taking the total variation of the constaint, we have

$$\delta\phi = \frac{\delta\phi}{\delta x_1}\delta x_1 + \frac{\delta\phi}{\delta x_2}dx_2 + \dots + \frac{d\phi}{dx_n}\delta x_n = 0 \quad \text{--- (5)}$$

Multiplying equation (4) by an unknown Lagrange multiplier, & subtracting it from equation (2) we have,

$$\left(\frac{\delta v}{\delta x_1} - \frac{\delta\phi}{\delta x_1}\right)\delta x_1 + \left(\frac{\delta v}{\delta x_2} - \lambda\frac{\delta\phi}{\delta x_2}\right)\delta x_2 + \dots +$$

$$\left(\frac{\delta v}{\delta x_n} - \lambda\frac{\delta\phi}{\delta x_n}\right)\delta x_n = 0 \text{ ---}$$

equation (5) is analougous to equation (56) results. as before, mathematical shorthand may be used to augment the potential energy, so that it assumes the form.

— (8)

$$V^* = V - \lambda\phi.$$

Then, the variation is taken as useal the Lagrange multiplier, λ, as assu to be constant, consider the following simple example: given the simple pendulum sh in Figure 6, determine the reaction on the Pivot, f_p, as well as th conditions of equilibrium using th Lagrange multiplier method.

Result.

1) whereas elementry calcules i
about small chages in the valu
of function without changes if
the function it self calculus of
variation is about infinitesimal
small changes in the function
self which are called variati

11) for a sufficient condition
see section variation & suf
condition for a minimum.

References.

1) courant & Hilbert 1953 P
11) wikkinpedia
3) google search.