Page No.
Date 21 06 2

Shri Krishna Mahavidyaly, Gunjoti

M.sc (Math) Sem - IV (2021-22)

Name - Vaishnavi Munishwar Ghodke

Project Name - Mechanics.

Project title - Lagrange's eq^n & Application.

PRN - 2020015200787661

Center no. - GUNJOTI (607)

15/20

Head
Department of Mathematics,
Shrikrishna Mahavidyalaya, Gunjoti
Tq. Omerga Dist. Osmanabad
(M.S.)-413606

D' Alembert's Principle & Lagrange's eq^n.

A virtual displacement of a system refers to a change in the configuration of the system as the result of any arbitrary infinitesimal change of the co-ordinates δr_i, consistent with the forces & constraints imposed on the system at the given instant t. The displacement is called virtual to distinguish it from an actual displacement of the system occurring in a time interval dt, during which the forces & constraints may be changing. Suppose the system is in equilibrium; i.e the total force on each particle vanishes, $F_i = 0$.

Then clearly the dot product $F_i \cdot \delta r_i$, which is the virtual work of the force F_i in the displacement δr_i, also vanishes. the sum of these vanishing products over all particles must likewise be zero.

$$\sum_i F_i \cdot \delta r_i = 0. \quad \text{——} \; (1)$$

As yet nothing has been said that has any new physical content. Decompose F_i, into the applied force, $F_i^{(a)}$, & the force of constraint f_i,

$$F_i = F_i^{(a)} + f_i, \quad \text{——} \; (2)$$

so that, eq^n (1) becomes,

$$\sum_i F_i^{(a)} \cdot \delta r_i + \sum_i f_i \cdot \delta r_i = 0 \quad \text{——} \ (3)$$

we now restrict ourselves to systems for wh
the net virtual work of the forces of constraint
zero.

we therefore have as the condition for equili
of a system that the virtual work of the applied
vanishes.

$$\sum_i F_i^{(a)} \cdot \delta r_i = 0 \quad \text{——} \ (4)$$

Eq^n (4) is often called the principle of
work.

<u>Note that</u>, the co-eficient of δr_i are not completel
independent but are connected by the constraints. In
to equate the co-efficients to zero, we must transform
principle into involving the virtual displacement o
which are independent. Eq^n (4) satisfies our needs
it does not contain f_i, but it deals only with s
we want involving the general motion of th
system.

To obtain such a principle, we use a device thought of by James Bernoulli & developed by D'Al the eq^t of motion,

$$F_i = \dot{P}_i ,$$

can be written as,

$$F_i - \dot{P}_i = 0$$

which states that the particles in the system will equilibrium under a force equal to the actual force a "reversed effective force" $-\dot{P}_i$. Instead of ① we ca immediately write,

$$\sum_i (F_i - \dot{P}_i) \cdot \delta r_i = 0 , \quad \text{——} \quad ⑤$$

& , making the same resolution into applied forces & constraint , there result,

$$\sum_i (F_i^{(a)} - \dot{P}_i) \cdot \delta r_i + \sum_i f_i \cdot \delta r_i = 0 .$$

we again restrict ourselves to systems for which the work of the forces of constraint & therefore obtai

$$\sum_i (F_i^{(a)} - \dot{P}_i) \cdot \delta r_i = 0 \quad \text{——} \quad ⑥$$

which is often called D' Alembert's principle.

The translation from r_i to q_i language sta
From the transformation eq^{n}

$$r_i = r_i (q_1, q_2, \text{-----} q_n, t) \quad \text{———} \quad (6)$$

(assumin n independent co-ordinates), & is carried o
means of the usual "chain rules" of the calcul
partial differentiation.

Thus, V_i is expressed in term of the
by the formula

$$V_i = \frac{dr_i}{dt} = \sum \frac{\partial r_i}{\partial q_k} \dot{q}_k + \frac{\partial r_i}{\partial t} \quad \text{———}$$

$||^{ly}$, the arbitary virtual displacement δr_i can be
connected with the virtual displacements δq_i by

$$\delta r_i = \sum_j \frac{\partial r_i}{\partial q_i} \delta q_j \quad \text{———} \quad (8)$$

In terms of the generalized co-ordinates, the virtual work of the F_i becomes.

$$\sum_i F_i \cdot \delta r_i = \sum_{i,j} F_i \cdot \frac{\partial r_i}{\partial q} \delta q_j$$

$$= \sum_j Q_j \, \delta q_j \quad \text{——— (9)}$$

where the Q_j are called the components of the generalized force, defined as

$$Q_j = \sum_i F_i \cdot \frac{\partial r_i}{\partial q_j} \quad \text{——— (10)}.$$

We turn now to the other term involved in eq^{t} (6) which may be written as

$$\sum_i \dot{P}_i \cdot \delta r_i = \sum_i m_i \ddot{r}_i \cdot \delta r_i .$$

Expressing δr_i by (8), in this becomes.

$$\sum_{i,j} m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j} \delta q_j$$

consider, now the relation.

$$\sum_i m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \dot{r}_i \cdot \frac{\partial r_i}{\partial q_j} \right) - m_i \dot{r}_i \cdot \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j} \right) \right] \quad \text{—} (11)$$

In last term of eq^n (11) we can interchange the differentiation w.r.t. t & q_j, for, in analogy — (7)

$$\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j} \right) = \frac{\partial \dot{r}_i}{\partial q_j} = \sum_k \frac{\partial^2 r_i}{\partial q_j \partial q_k} \dot{q}_k + \frac{\partial^2 r_i}{\partial q_j \partial t},$$

$$= \frac{\partial v_i}{\partial q_j}.$$

by eq^n (7). Further, we also see from eq^n (7) that.

$$\frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j} \quad \text{—} (12)$$

Substitution of these changes in (11) leads to the result that.

$$\sum_i m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} = \sum \left[\frac{d}{dt}\left(m_i v_i \cdot \frac{\partial v_i}{\partial \dot{q}_j} \right) - m_i v_i \cdot \frac{\partial v_i}{\partial q_j} \right],$$

& the second term on the left - hand side of eq^n (6) can be expanded into.

$$\sum_j \left\{ \frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}_j} \left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right] - \frac{\partial}{\partial q_j} \left(\sum_i \frac{1}{2} m_i v_i^2 \right) - Q_j \right\} \delta q_j$$

Identifying $\sum_i \frac{1}{2} m_i v_i^2$ with the system K. E. T. D' Alember principle (eq^n (6)) becomes.

$$\sum_j \left\{ \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] - Q_j \right\} \delta q_j = 0 \quad \text{--- (13)}$$

Any virtual displacement δq_j is then independent of δq_K & $\therefore$ the only way (13) to hold individual co-efficients to vanish.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \quad \text{—— (14)}$$

There are n such eq^t in all.

When the forces are derivable from a scalar potential funct V,

$$F_i = -\nabla_i V.$$

Then the generalized forces can be written as.

$$Q_j = \sum_i F_i \cdot \frac{\partial r_i}{\partial q_j} = -\sum_i \nabla_i V \cdot \frac{\partial r_i}{\partial q_j},$$

partial derivative of a funct $-V(r_1, r_2, \ldots r_N, t)$ w.r.t. q_j

$$Q_j = -\frac{\partial V}{\partial q_j} \quad \text{—— (15)}$$

see eq^t (8) eq^t (14) can then be rewritten as.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0. \quad \text{—— (16)}$$

Hence, we can include a term in V in the partial derivative w.r.t. q_j.

$$\frac{d}{dt}\left(\frac{\partial(T-V)}{\partial \dot{q}_j}\right) - \frac{\partial(T-V)}{\partial q_j} = 0.$$

or, defining a new funct^n, the Lagragian L, as.

$$L = T - V, \quad \text{——} \ (17)$$

the eq^n (14) become.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \quad \text{——} \ (18)$$

expression refferred to as "Lagrange's eq^n."

Simple Applications of the Lagrangian Formulation.

The previous section show that for system where we can define a Lagrangian.

Thus, T is given in general by,

$$T = \sum_i \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i \left(\sum_j \frac{\partial r_i}{\partial q_j} \dot{q}_j + \frac{\partial r_i}{\partial t} \right)^2$$

The expression for T is generalized co-ordinates
ll have the Form.

$$T = M_0 + \sum_j M_j \dot{q}_j + \frac{1}{2} \sum_{j,k} M_{jk} \dot{q}_j \dot{q}_k \quad \text{—— (1)}$$

where M_0, M_j, M_{jk} are definite funct of r's & t &
e of q's & t. show that.

$$M_0 = \sum_i \frac{1}{2} m_i \left(\frac{\partial r_i}{\partial t} \right)^2$$

$$M_j = \sum_i m_i \frac{\partial r_i}{\partial t} \cdot \frac{\partial r_i}{\partial q_j} \quad \text{—— (2)}$$

&

$$M_{jk} = \sum_i m_i \frac{\partial r_i}{\partial q_j} \cdot \frac{\partial r_i}{\partial q_k}$$

hus, the K.E. of system can always written as sum of
hree homogeneous funct of generalized velocities.

$$T = T_0 + T_1 + T_2 \quad \text{—— (3)}$$

simple example of procedure.

) single particle in space.
a) cartesian co-ordinate.
b) plan polar co-ordinate.

2) Atwood's machine
3) Time - dependent constraint - bead sliding on rotating

a) Motion of one particle : using Cartesian co-ordinates.

The generalized forces needed F_x, F_y & F_z then

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial y} = \frac{\partial T}{\partial z} = 0.$$

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x} \, , \quad \frac{\partial T}{\partial \dot{y}} = m\dot{y} \quad \frac{\partial T}{\partial \dot{z}} = m\dot{z} \, ,$$

& the eq^n of motion are,

$$\frac{d}{dt}(m\dot{x}) = F_x , \quad \frac{d}{dt}(m\dot{y}) = F_y , \quad \frac{d}{dt}(m\dot{z}) = F \qquad \text{--- (4)}$$

thus led back to the original Newton's eq^n of motion.

Reference -

H. Goldstein, Charles Poole, John safko : Classical Mechanics, Person 3rd Education, 2002.