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msc Mathematics sem - IV project

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★ Introduction.

- In mathematics, especially algebraic geometry the decomposition theorem is a set of results concerning the cohomology of algebraic varieties.

* statement

* Decomposition for smooth proper map.

The first case of the decomposition theorem arises via the hard lefschetz theorem which gives isomorphisms, for a smooth proper map $f: X \to Y$ of relative dimension d between two projective varieties

$$-\cup \eta^i : R^{d-i} f_*(\mathbb{Q}) \xrightarrow{\simeq} R^{d+i} f_*(\mathbb{Q}).$$

Here η is the fundamental class of a hyperplane section, f_* is the direct image (Pushforward) & $R^n f_*$ is the n-th derived functor of the direct image. This derived functor measures the n-th cohomologies of $f^{-1}(U)$, for $U \subset Y$. In fact, the particular case when Y is a point, amounts to the isomorphism.

$$-\cup \eta^i : H^{d-i}(X, \mathbb{Q}) \xrightarrow{\simeq} H^{d+i}(X, \mathbb{Q}).$$

This hard lefschetz isomorphism induces canonical isomorphisms.

$$Rf_*(\mathbb{Q}) \xrightarrow{\simeq} \bigoplus_{i=-d}^{d} R^{d+i} f_*(\mathbb{Q})[-d-i].$$

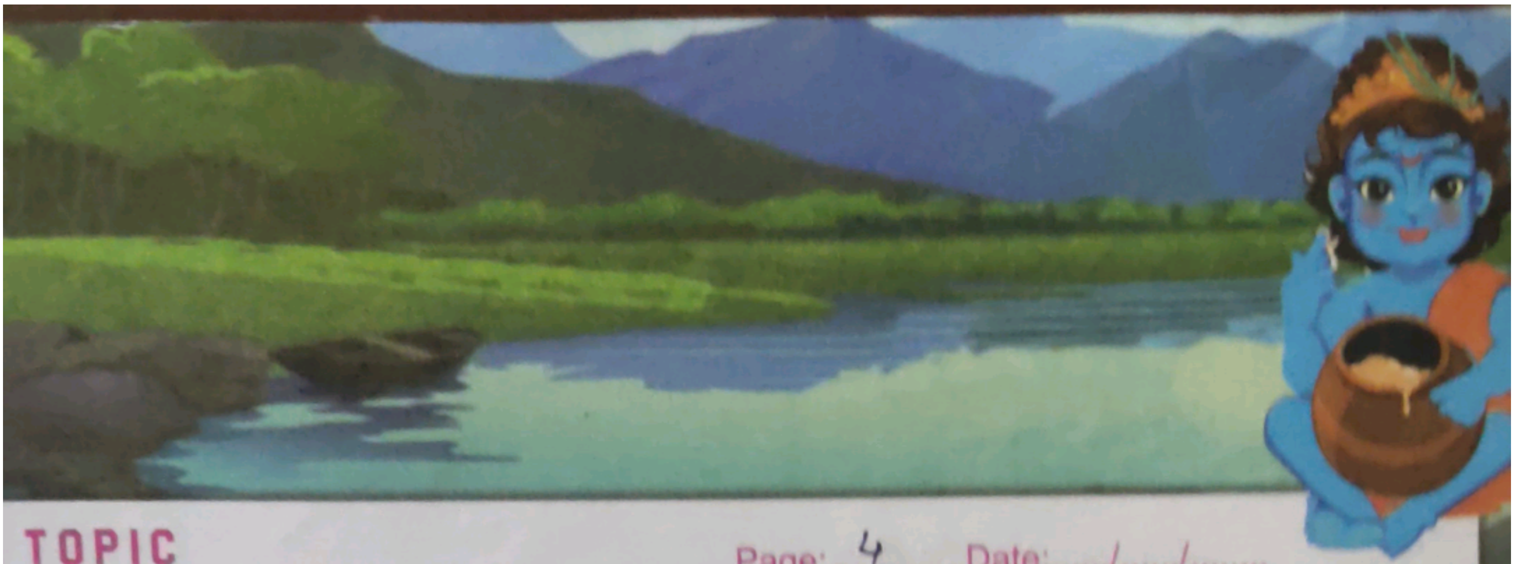
* Decomposition for proper maps

The decomposition theorem generallees this fact to the case of a proper, but not necessarily. smooth map $f: X \to Y$ betn varieties
In a nutshell, the results above remain true when the notion of local systems is replaced by perverse sheaves.

The hard lefschetz theorem above takes the following form: there is an isomorphism in the derived category of sheaves on Y:

$$ {}^{p}H^{-i}(Rf_* \mathbb{Q}) \cong {}^{p}H^{+i}(Rf_* \mathbb{Q}), $$

where Rf_* is the total derived function of f_* & . ${}^{p}H^{-i}$ is the i-th function with respect to the perverse t-structure. Moreover, there is an isomorphism



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$$Rf_* IC^\bullet_X \cong \bigoplus_i {}^pH^i(Rf_* IC^\bullet_X)[-i].$$

Where the summands are semi-simple perverseshoaves, meaning they are direct sums of pushforword of intersection cohomology sheaves.

If X is not smooth, then the above results remain true when $\mathbb{Q}[\dim X]$ is replaced by the intersection cohomology complex IC.

* Proofs:-

The decomposition theorem was first prov[ed] by Beilinson, Bernstein, & Deligue. Their proof is based on the usage of weights on l-adic sheaves in positive characteristic. A different proof using mixed Hodge modules was given by Saito. A more geometric proof, based on the notion of semismall maps was given by de cataldo & migliorini.

For semismall maps, the decomposition theorem also applies to chow motives

* Applications of ~~a~~ the theorem

* cohomology of a Rational Lefschetz pencil

consider a rational morphism $f : X \to P'$ from a smooth quasi-projective variety given by $[f_1(x) : f_2(x)]$. If we set the vanishing locus of $f_1 f_2$ as Y then there is an induced morphism $\tilde{X} = Bl_Y(X) \to P'$. We can compute the cohomology of X from the intersection cohomology of $Bl_Y(X)$ & subtracting off the cohomology from the blowup along Y. This can be done using the perverse ~~spectral~~ sequence

$$E_2^{lm} = H^l(P' ; {}^pH^m(IC^{\bullet}_{\tilde{X}}(\mathbb{Q})) \to$$

$$\Longrightarrow IH^{l+m}(\tilde{X};\mathbb{Q}) \simeq H^{l+m}(X;\mathbb{Q})$$

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★ References

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